Additional Examples of Chapter 14: Multirate Filter Banks and Wavelets

Example E14.1: Using the method of Eq. (14.13) develop a two-band polyphase decomposition of the following IIR transfer function

$$H(z)=\frac{2+3.1z^{-1}+1.5z^{-2}}{1+0.9z^{-1}+0.8z^{-2}}.$$

Answer: $H(zW_2^1)=H(-z)=\dfrac{2-3.1z^{-1}+1.5z^{-2}}{1-0.9z^{-1}+0.8z^{-2}}$. Thus,

$$E_0(z^2)=\tfrac{1}{2}[H(z)+H(-z)]=\frac{1}{2}\left[\frac{2+3.1z^{-1}+1.5z^{-2}}{1+0.9z^{-1}+0.8z^{-2}}+\frac{2-3.1z^{-1}+1.5z^{-2}}{1-0.9z^{-1}+0.8z^{-2}}\right]$$

$$=\frac{4+0.62z^{-2}+2.4z^{-4}}{1+0.79z^{-2}+0.64z^{-4}}.$$

$$z^{-1}E_1(z^2)=\tfrac{1}{2}[H(z)-H(-z)]=\frac{1}{2}\left[\frac{2+3.1z^{-1}+1.5z^{-2}}{1+0.9z^{-1}+0.8z^{-2}}-\frac{2-3.1z^{-1}+1.5z^{-2}}{1-0.9z^{-1}+0.8z^{-2}}\right]$$

$=\dfrac{2.6z^{-1}+2.26z^{-3}}{1+0.79z^{-2}+0.64z^{-4}}$. Hence, a two-band polyphase decomposition of $H_2(z)$ is given by

$$H(z)=\left(\frac{4+0.62z^{-2}+2.4z^{-4}}{1+0.79z^{-2}+0.64z^{-4}}\right)+z^{-1}\left(\frac{2.6+2.26z^{-2}}{1+0.79z^{-2}+0.64z^{-4}}\right).$$

Example E14.2: Using the method of Eq. (14.13) develop a three-band polyphase decomposition of the IIR transfer function of Example E14.1.

Answer: $\begin{bmatrix} H(z) \\ H(W_3^1 z) \\ H(W_3^2 z)\end{bmatrix}=\begin{bmatrix} 1 & 1 & 1 \\ 1 & W_3^{-1} & W_3^{-2} \\ 1 & W_3^{-2} & W_3^{-1}\end{bmatrix}\begin{bmatrix} E_0(z^3) \\ z^{-1}E_1(z^3) \\ z^{-2}E_2(z^3)\end{bmatrix}$ or

$\begin{bmatrix} E_0(z^3) \\ z^{-1}E_1(z^3) \\ z^{-2}E_2(z^3)\end{bmatrix}=\dfrac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & W_3^{1} & W_3^{2} \\ 1 & W_3^{2} & W_3^{1}\end{bmatrix}\begin{bmatrix} H(z) \\ H(W_3^1 z) \\ H(W_3^2 z)\end{bmatrix}$. Thus, $E_0(z^3)=\frac{1}{3}\left[H(z)+H(zW_3^1)+H(zW_3^2)\right]$

$$=\frac{1}{3}\left[\frac{2+3.1z^{-1}+1.5z^{-2}}{1+0.9z^{-1}+0.8z^{-2}}+\frac{2+3.1e^{j2\pi/3}z^{-1}+1.5e^{j4\pi/3}z^{-2}}{1+0.9e^{j2\pi/3}z^{-1}+0.8e^{j4\pi/3}z^{-2}}+\frac{2+3.1e^{j4\pi/3}z^{-1}+1.5e^{j2\pi/3}z^{-2}}{1+0.9e^{j4\pi/3}z^{-1}+0.8e^{j2\pi/3}z^{-2}}\right]$$

$$=\frac{2-2.759z^{-3}+0.96z^{-6}}{1-1.431z^{-3}+0.512z^{-6}}.$$

$$z^{-1}E_1(z^3)=\tfrac{1}{3}\left[H(z)+W_3^1H(zW_3^1)+W_3^2H(zW_3^2)\right]$$

$$= \frac{1}{3}\Big[\frac{2+3.1z^{-1}+1.5z^{-2}}{1+0.9z^{-1}+0.8z^{-2}} + e^{j2\pi/3}\frac{2+3.1e^{j2\pi/3}z^{-1}+1.5e^{j4\pi/3}z^{-2}}{1+0.9e^{j2\pi/3}z^{-1}+0.8e^{j4\pi/3}z^{-2}}$$

$$+ e^{j4\pi/3}\frac{2+3.1e^{j4\pi/3}z^{?}+1.5e^{j2\pi/3}z^{?}}{1+0.9e^{j4\pi/3}z^{?}+0.8e^{j2\pi/3}z^{?}}\Big] = z^{-1}\left(\frac{1.3-0.937z^{?}}{1-1.431z^{?}+0.512z^{?}}\right),$$

$$z^{-2}E_2(z^3) = \frac{1}{3}\Big[H(z) + W_3^2\,H(zW_3^1) + W_3^1\,H(zW_3^2)\Big]$$

$$= \frac{1}{3}\Big[\frac{2+3.1z^{-1}+1.5z^{-2}}{1+0.9z^{-1}+0.8z^{-2}} + e^{j4\pi/3}\frac{2+3.1e^{j2\pi/3}z^{-1}+1.5e^{j4\pi/3}z^{-2}}{1+0.9e^{j2\pi/3}z^{-1}+0.8e^{j4\pi/3}z^{-2}}$$

$$+ e^{j2\pi/3}\frac{2+3.1e^{j4\pi/3}z^{?}+1.5e^{j2\pi/3}z^{?}}{1+0.9e^{j4\pi/3}z^{?}+0.8e^{j2\pi/3}z^{?}}\Big] = z^{-2}\left(\frac{-1.27+0.904z^{?}}{1-1.431z^{?}+0.512z^{?}}\right).$$ Hence,

$$E_0(z) = \frac{2-2.759z^{-1}+0.96z^{-2}}{1-1.431z^{-1}+0.512z^{-2}}\quad E_1(z) = \frac{1.3-0.937z^{?}}{1-1.431z^{?}+0.512z^{?}},$$ and

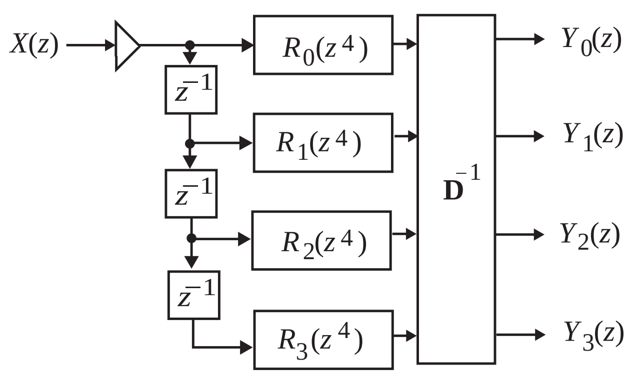
$$E_2(z) = \frac{-1.27+0.904z^{?}}{1-1.431z^{?}+0.512z^{?}}.$$

Example E14.3: The four-channel analysis filter bank of Figure E14.1(a), where **D** is a 4×4 DFT matrix, is characterized by the set of four transfer functions: $H_i(z(=Y_i(z)/X(z)$, $i = 0,1,2,3$.
Let the transfer functions of the four subfilters be given by

$$E_0(z) = 1+0.3z^{-1}-0.8z^{-2},\ E_1(z) = 2-1.5z^{-1}+3.1z^{-2},$$
$$E_2(z) = 4-0.9z^{-1}+2.3z^{-2},\ E_4(z) = 1+3.7z^{-1}+1.7z^{-2}.$$

(a) Determine the expressions for the four transfer functions $H_0(z),\ H_1(z),\ H_2(z),$ and $H_3(z)$.

(b) Assume that the analysis filter $H_2(z)$ has a magnitude response as indicated in Figure E14.1(b). Sketch the magnitude responses of the other three analysis filters.



(a)

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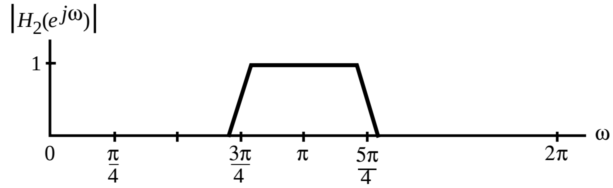


(b)

Figure E14.1

Answer: (a) $\begin{bmatrix} H_0(z) \\ H_1(z) \\ H_2(z) \\ H_3(z) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} E_0(z^4) \\ z^{-1}E_1(z^4) \\ z^{-2}E_2(z^4) \\ z^{-3}E_3(z^4) \end{bmatrix}$. Hence,

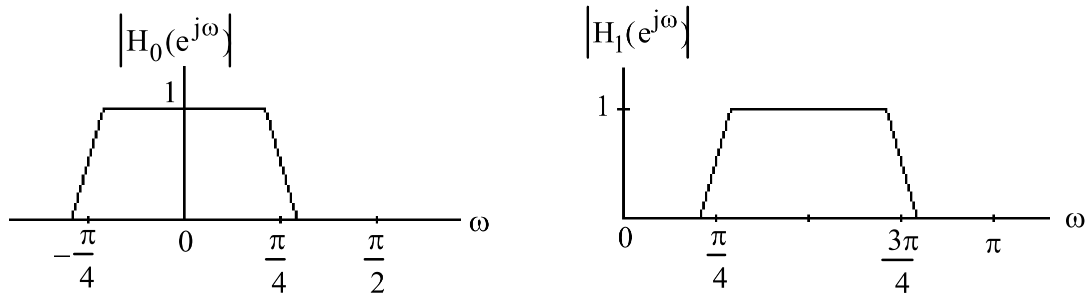
$$H_0(z) = E_0(z^4) + z^{-1}E_1(z^4) + z^{-2}E_2(z^4) + z^{-3}E_3(z^4)$$
$$= 1 + 2z^{-1} + 4z^{-2} + z^{-3} + 0.3z^{-4} - 1.5z^{-5} - 0.9z^{-6} + 3.7z^{-7} - 0.8z^{-8} + 3.1z^{-9} + 2.3z^{-10} + 1.7z^{-11}$$

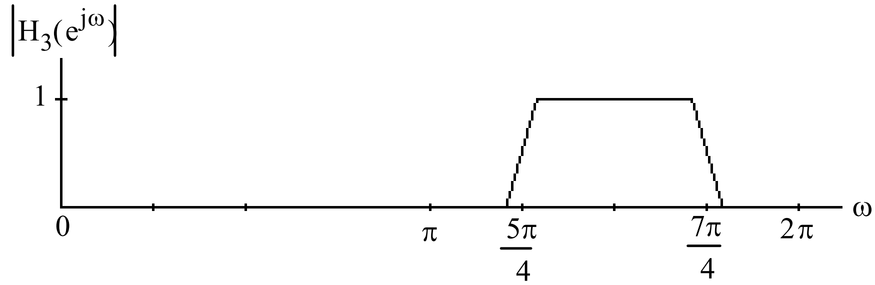
$$H_1(z) = E_0(z^4) - jz^{-1}E_1(z^4) - z^{-2}E_2(z^4) + jz^{-3}E_3(z^4) = 1 - j2z^{-1} - 4z^{-2}$$
$$+ jz^{-3} + 0.3z^{-4} + j1.5z^{-5} - 0.9z^{-6} + j3.7z^{-7} - 0.8z^{-8} - j3.1z^{-9} + 2.3z^{-10} + j1.7z^{-11},$$

$$H_2(z) = E_0(z^4) - z^{-1}E_1(z^4) + z^{-2}E_2(z^4) - z^{-3}E_3(z^4) = 1 - 2z^{-1} + 4z^{-2}$$
$$- z^{-3} + 0.3z^{-4} + 1.5z^{-5} - 0.9z^{-6} - 3.7z^{-7} - 0.8z^{-8} - 3.1z^{-9} + 2.3z^{-10} - 1.7z^{-11},$$

$$H_3(z) = E_0(z^4) + jz^{-1}E_1(z^4) - z^{-2}E_2(z^4) - jz^{-3}E_3(z^4) = 1 + j2z^{-1} - 4z^{-2}$$
$$- jz^{-3} + 0.3z^{-4} - j1.5z^{-5} + 0.9z^{-6} - j3.7z^{-7} + 0.8z^{-8} + j3.1z^{-9} - 2.3z^{-10} - j1.7z^{-11}$$

(b)





Example E14.4: (a) Decompose the causal third-order transfer function

$$G(z) = \frac{(1+z^{-1})^3}{6+2z^{-2}},$$

in the form

$$G(z) = \frac{1}{2}\{A_0(z) + A_1(z)\},$$

where $A_0(z)$ and $A_1(z)$ are stable allpass transfer functions.

(b) Realize G(z) as a parallel connection of allpass filters with $A_0(z)$ and $A_1(z)$ realized with the fewest number of multipliers.

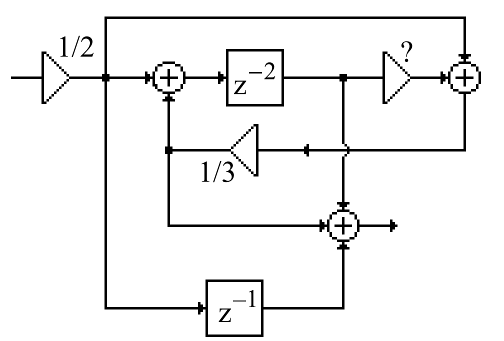
(c) Determine the transfer function H(z) which is power-complementary to G(z).

(d) Sketch the magnitude responses of G(z) and H(z).

Answer: (a) $G(z) = \dfrac{1+3z^{-1}+3z^{-2}+z^{-3}}{6+2z^{-2}} = \dfrac{1}{2}\left(\dfrac{1+3z^{-2}}{3+z^{-2}} + \dfrac{3z^{-1}+z^{-3}}{3+z^{-2}}\right) = \dfrac{1}{2}\left(\dfrac{1+3z^{-2}}{3+z^{-2}} + z^{-1}\right)$

$= \frac{1}{2}\left(A_0(z^2) + z^{-1}A_1(z^2)\right)$ where $A_0(z) = \dfrac{1+3z^{-1}}{3+z^{-1}}$, and $A_1(z) = z^{-1}$.

(b)



(c)

$$H(z) = \frac{1}{2}\left(A_0(z^2) - z^{-1}A_1(z^2)\right) = \frac{1}{2}\left(\frac{1+3z^{-2}}{3+z^{-2}} - z^{-1}\right) = \frac{1}{2}\left(\frac{1-3z^{-1}+3z^{-2}-z^{-3}}{3+z^{-2}}\right) = \frac{(1-z^{-1})^3}{6+2z^{-2}}.$$

(d)

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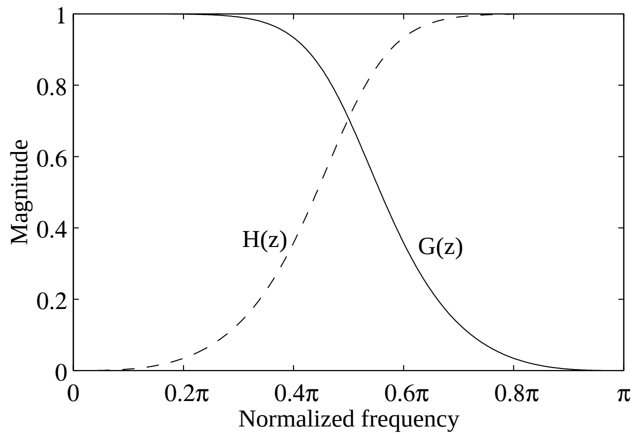

Example E14.5: Consider the QMF bank structure of Figure 14.20 with L = 4. Let the Type I polyphase component matrix be given by

$$\mathbf{E}(z) = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 13 & 9 & 7 \\ 3 & 9 & 11 & 10 \\ 2 & 7 & 10 & 15 \end{bmatrix}.$$

Determine the Type II polyphase component matrix $\mathbf{R}(z)$ such that the four-channel QMF structure is a perfect reconstruction system with an input-output relation $y[n] = 3x[n-3]$.

Answer: For $y[n] = 3x[n-3]$, we require $\mathbf{R}(z)\mathbf{E}(z) = 3\mathbf{I}$ or

$$\mathbf{R}(z) = 3\mathbf{E}^{-1}(z) = 3\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 13 & 9 & 7 \\ 3 & 9 & 11 & 10 \\ 2 & 7 & 10 & 15 \end{bmatrix}^{-1} = \begin{bmatrix} -3.8333 & -1.5 & 4.8333 & -2.333 \\ -1.5833 & 0.25 & 0.5833 & -0.333 \\ 4.5 & 0.5 & -2.5 & 1.0 \\ -1.75 & -0.25 & 0.75 & 0 \end{bmatrix}.$$

Example E14.6: Design a three-channel perfect reconstruction QMF bank whose analysis filters are given by

$$\begin{bmatrix} H_0(z) \\ H_1(z) \\ H_0(z) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ z^{-1} \\ z^{-2} \end{bmatrix}.$$

Develop a computationally efficient realization of the filter bank.

Answer: $\mathbf{E}(z) = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix}$. For perfect reconstruction,

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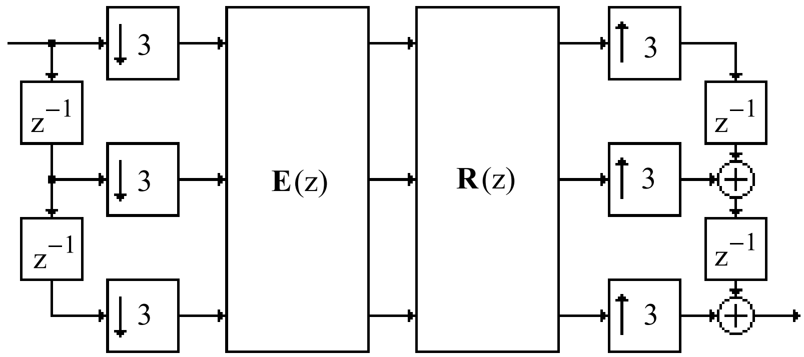
$\mathbf{R}(z) = \mathbf{E}^{-1}(z) = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0.5 & 1.5 & -2.5 \\ -0.5 & -0.5 & 1.5 \\ 0.5 & -0.5 & 0.5 \end{bmatrix}$. A computationally efficient realization of the filter bank is shown below:



Example E14.7: Design a three-channel perfect reconstruction QMF bank whose synthesis filters are given by

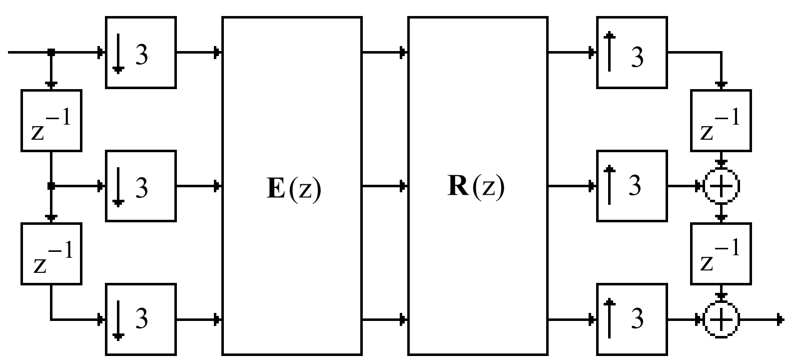
$$\begin{bmatrix} G_0(z) \\ G_1(z) \\ G_0(z) \end{bmatrix} = \begin{bmatrix} 4 & 2 & 3 \\ 5 & 4 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ z^{-1} \\ z^{-2} \end{bmatrix}.$$

Develop a computationally efficient realization of the filter bank.

Answer: For perfect reconstruction we require

$\mathbf{R}(z) = \mathbf{E}^{-1}(z) = \begin{bmatrix} 4 & 2 & 3 \\ 5 & 4 & 1 \\ 2 & 1 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 1.2222 & -0.3333 & -1.1111 \\ -1.4444 & 0.6667 & 1.2222 \\ -0.3333 & 0 & 0.6667 \end{bmatrix}$. A computationally

efficient realization of the filter bank is shown below:



Example E14.8: Consider the power-symmetric FIR transfer function

$$H_0(z) = \frac{1}{2} - z^{-1} + \frac{21}{2} z^{-2} - \frac{27}{2} z^{-3} - 5z^{-4} - \frac{5}{2} z^{-5}.$$

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Using $H_0(z)$ as one of the analysis filter, determine the remaining three filters of the corresponding two-channel orthogonal filter bank. Show that the filter bank is alias-free and satisfies the perfect reconstruction condition.

Answer: $H_0(z) = \frac{1}{2} - z^{-1} + \frac{21}{2} z^{-2} - \frac{27}{2} z^{-3} - 5z^{-4} - \frac{5}{2} z^{-5}$.

$H_0(z^{-1}) = \frac{1}{2} - z + \frac{21}{2} z^{2} - \frac{27}{2} z^{3} - 5z^{4} - \frac{5}{2} z^{5}$.

The highpass analysis filter is given by

$H_1(z) = z^{-5} H_0(-z^{-1}) = \frac{5}{2} - 5z^{-1} + \frac{27}{2} z^{-2} + \frac{21}{2} z^{-3} + z^{-4} + \frac{1}{2} z^{-5}$.

The two synthesis filters are time-reversed versions of the analysis filter as per Eq. (14.92) and are given by

$F_0(z) = z^{-5} H_0(z^{-1}) = -2.5 - 5z^{-1} - 13.5z^{-2} + 10.5z^{-3} - z^{-4} + 0.5z^{-5}$, and

$F_1(z) = z^{-5} H_1(z^{-1}) = 0.5 + z^{-1} + 10.5z^{-2} + 13.5z^{-3} - 5z^{-4} + 2.5z^{-5}$.
