Additional Examples of Chapter 3: Discrete-Time Signals and Systems in the Frequency Domain

Example E3.1: Determine the DTFT $X(e^{j\omega})$ of the causal sequence

$$x[n] = A\alpha^n \cos(\omega_o n + \phi)\mu[n],$$

where A, α, ω_o, and ϕ are real.

Answer: $x[n] = \alpha^n \cos(\omega_o n + \phi)\mu[n] = A\alpha^n \left(\frac{e^{j\omega_0 n} e^{j\phi} + e^{-j\omega_0 n} e^{-j\phi}}{2} \right) \mu[n]$

$= \frac{A}{2} e^{j\phi} \left(\alpha e^{j\omega_o} \right)^n \mu[n] + \frac{A}{2} e^{-j\phi} \left(\alpha e^{-j\omega_o} \right) \mu[n]$. Therefore,

$$X(e^{j\omega}) = \frac{A}{2} e^{j\phi} \frac{1}{1 - \alpha e^{-j\omega} e^{j\omega_o}} + \frac{A}{2} e^{-j\phi} \frac{1}{1 - \alpha e^{-j\omega} e^{-j\omega_o}}.$$

Example E3.2: Determine the inverse DTFT h[n] of

$$H(e^{j\omega}) = (3 + 2\cos\omega + 4\cos 2\omega)\cos(\omega/2) e^{-j\omega/2}.$$

Answer: $H(e^{j\omega}) = \left[3 + 2\left(\frac{e^{j\omega} + e^{-j\omega}}{2} \right) + 4\left(\frac{e^{j2\omega} + e^{-j2\omega}}{2} \right) \right] \cdot \left(\frac{e^{j\omega/2} + e^{-j\omega/2}}{2} \right) \cdot e^{-j\omega/2}$

$= \frac{1}{2} \left(2e^{j2\omega} + 3e^{j\omega} + 4 + 4e^{-j\omega} + 3e^{-j2\omega} + 2e^{-j3\omega} \right)$ Hence, the inverse of $H(e^{j\omega})$ is a length-6 sequence given by $h[n] = [1 \quad 1.5 \quad 2 \quad 2 \quad 1.5 \quad 1], \ -2 \le n \le 3.$

Example E3.3: Let $X(e^{j\omega})$ denote the DTFT of a real sequence x[n]. Determine the inverse DTFT y[n] of $Y(e^{j\omega}) = X(e^{j3\omega})$ in terms of x[n].

Answer: $Y(e^{j\omega}) = X(e^{j3\omega}) = X\left((e^{j\omega})^3 \right)$ Now, $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$. Hence,
$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y[n] e^{-j\omega n} = X\left((e^{j\omega})^3 \right) = \sum_{n=-\infty}^{\infty} x[n] (e^{-j\omega n})^3 = \sum_{m=-\infty}^{\infty} x[m/3] e^{-j\omega m}$.

Therefore, $y[n] = \begin{cases} x[n], & n = 0, \pm 3, \pm 6, \ldots \\ 0, & \text{otherwise.} \end{cases}$

Example E3.4: Without computing the DTFT, determine whether the following DTFT has an inverse that is even or an odd sequence:

$$x[n] = \begin{cases} n^3, & -N \le n \le N, \\ 0, & \text{otherwise.} \end{cases}$$

Answer: Since $(-n)^3 = -n^3$, x[n] is an odd sequence with an imaginary-valued DTFT.

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Example E3.5: Let $X(e^{j\omega})$ denote the DTFT of a real sequence x[n]. Determine the DTFT $Y(e^{j\omega})$ of the sequence $y[n] = x[n] \circledast x[-n]$.

Answer: Let $u[n] = x[-n]$, and let $X(e^{j\omega})$ and $U(e^{j\omega})$ denote the DTFTs of x[n] and u[n], respectively. From the convolution property of the DTFT given in Table 3.4, the DTFT of $y[n] = x[n] \circledast u[n]$ is given by $Y(e^{j\omega}) = X(e^{j\omega})\, U(e^{j\omega})$. From Table 3.4, $U(e^{j\omega}) = X(e^{-j\omega})$. But from Table 3.2, $X(e^{-j\omega}) = X^*(e^{j\omega})$. Hence, $Y(e^{j\omega}) = X(e^{j\omega})\, X^*(e^{j\omega}) = \left|X(e^{j\omega})\right|^2$ which is real-valued function of ω.

Example E3.6: Without computing the inverse DTFT, determine whether the inverse of the DTFT shown in Figure E3.1 is an even or an odd sequence.

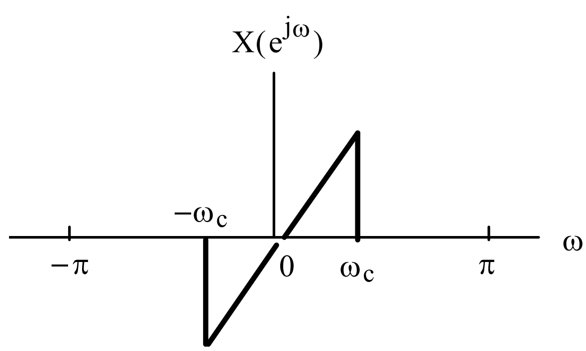

Figure E3.1

Answer: $X(e^{j\omega})$ is a real-valued function of ω. Hence, its inverse is an even sequence.

Example E3.7: A sequence x[n] has zero-phase DTFT as shown in Figure E3.2 Sketch the DTFT of the sequence $x[n]e^{-j\pi n/3}$.

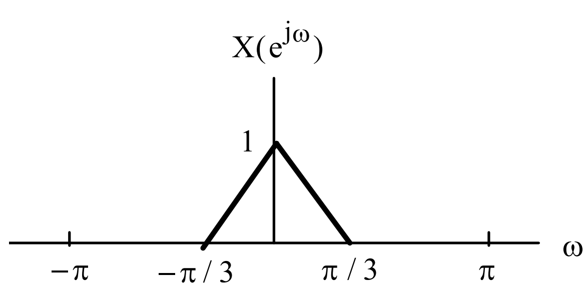

Figure E3.2

Answer: From the frequency-shifting property of the DTFT given in Table 3.4, the DTFT of $x[n]e^{-j\pi n/3}$ is given by $X(e^{j(\omega+\pi/3)})$. A sketch of this DTFT is shown below.

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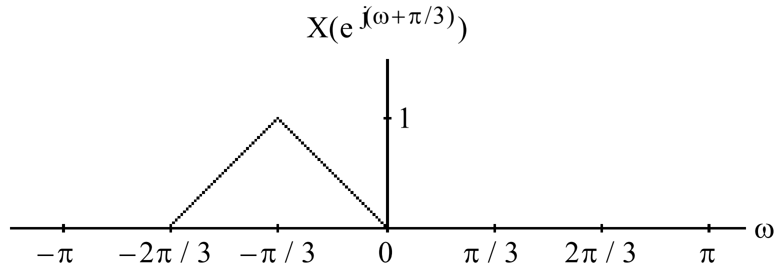

Example E3.8: Consider $\{x[n]\} = \{3 \quad 0 \quad 1 \quad -2 \quad -3 \quad 4 \quad 1 \quad 0 \quad -1\}$, $-3 \le n \le 5$, with a DTFT given by $X(e^{j\omega})$. Evaluate the following functions of $X(e^{j\omega})$ without computing the transform itself: (a) $X(e^{j0})$, (b) $X(e^{j\pi})$, (c) $\int_{-\pi}^{\pi} X(e^{j\omega})d\omega$, (d) $\int_{-\pi}^{\pi} \left|X(e^{j\omega})\right|^2 d\omega$, and (e) $\int_{-\pi}^{\pi} \left|\frac{dX(e^{j\omega})}{d\omega}\right|^2 d\omega$.

Answer: (a) $X(e^{j0}) = \sum_{n=-\infty}^{\infty} x[n] = 3+1-2-3+4+1-1 = 3.$

(b) $X(e^{j\pi}) = \sum_{n=-\infty}^{\infty} x[n]e^{j\pi n} = -3-1-2+3+4-1+1 = 1.$

(c) $\int_{-\pi}^{\pi} X(e^{j\omega})d\omega = 2\pi x[0] = -4\pi.$

(d) $\int_{-\pi}^{\pi} \left|X(e^{j\omega})\right|^2 d\omega = 2\pi \sum_{n=-\infty}^{\infty} \left|x[n]\right|^2 = 82\pi.$ (Using Parseval's relation)

(e) $\int_{-\pi}^{\pi} \left|\frac{dX(e^{j\omega})}{d\omega}\right|^2 d\omega = 2\pi \sum_{n=-\infty}^{\infty} \left|n \cdot x[n]\right|^2 = 378\pi.$ (Using Parseval's relation with differentiation-in-frequency property)
