Additional Examples of Chapter 5: Finite-Length Discrete Transforms

Example E5.1: Consider the following length-8 sequences defined for $0 \le n \le 7$:

(a) $\{x_1[n]\} = \{1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 1 \quad 1\}$, (b) $\{x_2[n]\} = \{1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad -1 \quad -1\}$,

(c) $\{x_3[n]\} = \{0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0 \quad -1 \quad -1\}$, (d) $\{x_4[n]\} = \{0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 1 \quad 1\}$.

Answer: (a) $x_1[\langle -n\rangle_8] = [1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 1 \quad 1] = x_1[n]$. Thus, $x_1[n]$ is a periodic even sequence, and hence it has a real-valued 8-point DFT.

(b) $x_2[\langle -n\rangle_8] = [1 \quad -1 \quad -1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1]$. Thus, $x_2[n]$ is neither a periodic even or a periodic odd sequence. Hence, its 8-point DFT is a complex sequence.

(c) $x_3[\langle -n\rangle_8] = [0 \quad -1 \quad -1 \quad 0 \quad 0 \quad 0 \quad 1 \quad 1] = -x_3[n]$. Thus, $x_3[n]$ is a periodic odd sequence, and hence it has an imaginary-valued 8-point DFT.

(d) $x_4[\langle -n\rangle_8] = [0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 1 \quad 1] = x_4[n]$. Thus, $x_4[n]$ is a periodic even sequence, and hence it has a real-valued 8-point DFT.

Example E5.2: Let G[k] and H[k] denote the 7-point DFTs of two length-7 sequences, g[n] and h[n], $0 \le n \le 6$, respectively. If

$$G[k] = \{1 + j2, \quad -2 + j3, \quad -1 - j2, \quad 0, \quad 8 + j4, \quad -3 + j, \quad 2 + j5\}$$

and $h[n] = g[\langle n-3\rangle_7]$, determine H[k] without computing the IDFT g[n].

Answer: $H[k] = \mathrm{DFT}\{h[n]\} = \mathrm{DFT}\{g[\{n-3>_7]\} = W_7^{3k} G[k] = e^{-j\frac{6\pi k}{7}} G[k]$

$$= \left[1 + j2, \quad e^{-j\frac{6\pi}{7}}(-2 + j3), \quad e^{-j\frac{12\pi}{7}}(-1 - j2), \quad 0, \quad e^{-j\frac{24\pi}{7}}(8 + j4), \quad e^{-j\frac{30\pi}{7}}(-3 + j2), \quad e^{-j\frac{36\pi}{7}}(2 + j5)\right]$$

Example E5.3: Let G[k] and H[k] denote the 7-point DFTs of two length-7 sequences, g[n] and h[n], $0 \le n \le 6$, respectively. If $g[n] = \{-3.1, \quad 2.4, \quad 4.5, \quad -6, \quad 1, \quad -3, \quad 7\}$ and $G[k] = H[\langle k-4\rangle_7]$, determine h[n] without computing the DFT G[k].

Answer: $h[n] = \mathrm{IDFT}\{H[k]\} = \mathrm{IDFT}\{G[< k-4 >_7]\} = W_7^{-4n} g[n] = e^{j\frac{8\pi n}{7}} g[n]$

$$= \left[-3.1, \quad 2.4\,e^{j8\pi/7}, \quad 4.5\,e^{j16\pi/7}, \quad -6\,e^{j24\pi/7}, \quad e^{j32\pi/7}, \quad -3\,e^{j40\pi/7}, \quad 7\,e^{j42\pi/7}\right]$$

Example E5.4: Let X[k], $0 \le k \le 13$, be a 14-point DFT of a length-14 real sequence x[n]. The first 8 samples are given by X[0] = 12, X[1] = –1+j3, X[2] = 3+j4, X[3] = 1–j5, X[4] = –2+j2, X[5] = 6+j3, X[6] = –2–j3, X[7] = 10. Determine the remaining samples of X[k]. Evaluate the following functions of x[n] without computing the IDFT of X[k]:

(a) $x[0]$, (b) $x[7]$, (c) $\sum_{n=0}^{13} x[n]$, (d) $\sum_{n=0}^{13} e^{j(4\pi n/7)} x[n]$, and (e) $\sum_{n=0}^{13} |x[n]|^2$.

Answer: $X[8] = X^*[\langle -8\rangle_{14}] = X^*[6] = -2 + j3$, $X[9] = X^*[\langle -9\rangle_{14}] = X^*[5] = 6 - j3$,

$X[10] = X^*[\langle -10\rangle_{14}] = X^*[4] = -2 - j2$, $X[11] = X^*[\langle -11\rangle_{14}] = X^*[3] = 1 + j5$,

$X[12] = X^*[\langle -12\rangle_{14}] = X^*[2] = 3 - j4$, $X[13] = X^*[\langle -13\rangle_{14}] = X^*[1] = -1 - j3$.

(a) $x[0] = \frac{1}{14}\sum_{k=0}^{13} X[k] = \frac{32}{14} = 2.2857$,

(b) $x[7] = \frac{1}{14}\sum_{k=0}^{13} (-1)^k X[k] = -\frac{12}{14} = -0.8571$,

(c) $\sum_{n=0}^{13} x[n] = X[0] = 12$,

(d) Let $g[n] = e^{j(4\pi n/7)} x[n] = W_{14}^{-4n} x[n]$. Then $\text{DFT}\{g[n]\} = \text{DFT}\{W_{14}^{-4n} x[n]\} = X[\rangle k-4\rangle_{14}]$
$= \begin{bmatrix} X[10] & X[11] & X[12] & X[13] & X[0] & X[1] & X[2] & X[3] & X[4] & X[5] & X[6] & X[7] & X[8] & X[9] \end{bmatrix}$

Thus, $\sum_{n=0}^{13} g[n] = \sum_{n=0}^{13} e^{j(4\pi n/7)} x[n] = X[10] = -2 \ \square j2$,

(e) Using Parseval's relation, $\sum_{n=0}^{13} |x[n]|^2 = \frac{1}{14}\sum_{k=0}^{13} |X[k]|^2 = \frac{498}{14} = 35.5714$.

Example E5.5: Consider the sequence $x[n]$ defined for $0 \le n \le 11$,

$$\{x[n]\} = \{3, \ -1, \ 2, \ 4, \ -3, \ -2, \ 0, \ 1, \ -4, \ 6, \ 2, \ 5\},$$

with a 12-point DFT given by $X[k]$, $0 \le k \le 11$. Evaluate the following functions of $X[k]$ without computing the DFT:

(a) $X[0]$, (b) $X[6]$, (c) $\sum_{k=0}^{11} X[k]$, (d) $\sum_{k=0}^{11} e^{-j(2\pi k/3)} X[k]$, and (e) $\sum_{k=0}^{11} |X[k]|^2$.

Answer: (**a**) $X[0] = \sum_{n=0}^{11} x[n] = 13$, (b) $X[6] = \sum_{n=0}^{11} (-1)^n x[n] = -13$,

(c) $\sum_{k=0}^{11} X[k] = 12 \cdot x[0] = 36$, (d) The inverse DFT of $e^{-j(4\pi k/6)} X[k]$ is $x[\langle n-4\rangle_{12}]$. Thus,

$\sum_{k=0}^{11} e^{-j(4\pi k/6)} X[k] = 12 \cdot x[< 0 - 4 >_{12}] = 12 \cdot x[8] = -48$.

(e) From Parseval's relation, $\sum_{k=0}^{11} |X[k]|^2 = 12 \cdot \sum_{n=0}^{11} |x[n]|^2 = 1500$.

Additional Examples of Chapter 5: Finite-Length Discrete Transforms

Example E5.6: The even samples of the 11-point DFT of a length-11 real sequence are given by X[0] = 4, X[2] = –1 + j3, X[4] = 2 + j5, X[6] = 9 – j6, X[8] = –5 – j8, and X[10] = $\sqrt{3}$ – j2. Determine the missing odd samples.

Answer: Since x[n] is a real sequence, its DFT satisfies $X[k] = X^*[\langle -k \rangle_N]$ where N = 11 in this case. Therefore, $X[1] = X^*[\langle -1 \rangle_{11}] = X^*[10] = \sqrt{3} + j2$,
$X[3] = X^*[\langle -3 \rangle_{11}] = X^*[8] = -5 + j8$, $X[5] = X^*[\langle -5 \rangle_{11}] = X^*[6] = 9 + j6$,
$X[7] = X^*[\langle -7 \rangle_{11}] = X^*[4] = 2 - j5$, $X[9] = X^*[\langle -9 \rangle_{11}] = X^*[2] = -1 - j3$.

Example E5.7: The following 6 samples of the 11-point DFT X[k], $0 \le k \le 10$, are given: X[0] = 12, X[2] –3.2 – j2, X[3] = 5.3 – j4.1, X[5] = 6.5 + j9, X[7] = –4.1 + j0.2, and X[10] = –3.1 + j5.2. Determine the remaining 5 samples.

Answer: The N-point DFT X[k] of a length-N real sequence x[n] satisfy $X[k] = X^*[\langle -k \rangle_N]$. Here N = 11. Hence, the remaining 5 samples are $X[1] = X^*[\langle -1 \rangle_{11}] = X^*[10] = -3.1 - j5.2$,
$X[4] = X^*[\langle -4 \rangle_{11}] = X^*[7] = -4.1 - j0.2$, $X[6] = X^*[\langle -6 \rangle_{11}] = X^*[5] = 6.5 - j9$,
$X[8] = X^*[\langle -8 \rangle_{11}] = X^*[3] = 5.3 + j4.1$, $X[9] = X^*[\langle -9 \rangle_{11}] = X^*[2] = -3.2 + j2$.

Example E5.8: A length-10 sequence x[n], $0 \le n \le 9$, has a real-valued 10-point DFT X[k], $0 \le k \le 9$. The first 6 samples of x[n] are given by: x[0] = 2.5, x[1] = 0.7 – j0.08, x[2] = –3.25 + j1.12, x[3] = –2.1 + j4.6, x[4] = 2.87 + j2, and x[5] = 5. Determine the remaining 4 samples.

Answer: A length-N periodic even sequence x[n] satisfying $x[n] = x^*[\langle -n \rangle_N]$ has a real-valued N- point DFT X[k]. Here N = 10. Hence, the remaining 4 samples of x[n] are given by
$x[6] = x^*[\langle -6 \rangle_{10}] = x^*[4] = 2.87 - j2$, $x[7] = x^*[\langle -7 \rangle_{10}] = x^*[3] = -2.1 - j4.6$,
$x[8] = x^*[\langle -8 \rangle_{10}] = x^*[2] = -3.25 - j1.12$, and $x[9] = x^*[\langle -9 \rangle_{10}] = x^*[1] = 0.7 + j0.08$.

Example E5.9: The 8-point DFT of a length-8 complex-valued sequence v[n] = x[n] + jy[n] is given by

$$\{V[k]\} = \{-2 + j3,\quad 1 + j5,\quad -4 + j7,\quad 2 + j6,\quad -1 - j3,\quad 4 - j,\quad 3 + j8,\quad j6\}.$$

Without computing the IDFT of V[k], determine the 8-point DFTs X[k] and Y[k] of the real sequences x[n] and y[n], respectively.

Answer: $v[n] = x[n] + j\,y[n]$. Hence, $X[k] = \frac{1}{2}\{V[k] + V^*\langle -k \rangle_8]\}$ is the 8-point DFT of x[n],

and $Y[k] = \frac{1}{2j}\{V[k] - V^*\langle -k \rangle_8]\}$ is the 8-point DFT of y[n]. Now,

$$V^*[\langle -k \rangle_8] = [-2 - j3,\quad -j6,\quad 3 - j8,\quad 4 + j,\quad -1 + j3,\quad 2 - j6,\quad -4 - j7,\quad 1 - j5].$$

Therefore,

$$X[k] = [-0.2,\quad 0.5 - j0.5,\quad -0.5 - j0.5,\quad 3 + j3.5,\quad -1,\quad 3 - j3.5,\quad -0.5 + j0.5,\quad 0.5 + j0.5]$$

$$Y[k] = [3,\quad 5.5 - j0.5,\quad 7.5 + j3.5,\quad 2.5 + j\quad -3,\quad 2.5 - j,\quad 7.5 - j3.5,\quad 5.5 + j0.5]$$

Additional Examples of Chapter 5:
Finite-Length Discrete Transforms

Example E5.10: Determine the 4-point DFTs of the following length-4 sequences, $0 \le n \le 3$, defined for by computing a single DFT: $\{g[n]\} = \{-2, \quad 1, \quad -3, \quad 4\}$, $\{h[n]\} = \{1, \quad 2, \quad -3, \quad 2\}$.

Answer: $v[n] = g[n] + jh[n] = [-2 + j, \quad 1 + j2, \quad -3 - j3, \quad 4 + j2]$. Therefore,

$$\begin{bmatrix} V[0] \\ V[1] \\ V[2] \\ V[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} -2+j \\ 1+j2 \\ -3-j3 \\ 4+j2 \end{bmatrix} = \begin{bmatrix} j2 \\ 1+j7 \\ -10-j6 \\ 1+j \end{bmatrix}, \text{ i.e., } \{V[k]\} = [j2, \quad 1+j7, \quad -10-j6, \quad 1+j].$$

Thus, $\{V^*[\langle -k \rangle_4]\} = [-j2, \quad 1-j, \quad -10+j6, \quad 1-j7]$.

Therefore, $G[k] = \frac{1}{2}\{V[k] + V^*[\langle -k \rangle_4]\} = [0, \quad 1+j3, \quad -10, \quad 1-j3]$ and

$H[k] = \frac{1}{2j}\{V[k] - V^*[\langle -k \rangle_4]\} = [2, \quad 4, \quad -6, \quad 4]$.
