Additional Examples of Chapter 2: Discrete-Time Signals and Systems

Example E2.1: Analyze the block diagram of the LTI discrete-time system of Figure E2.1 and develop the relation between y[n] and x[n].

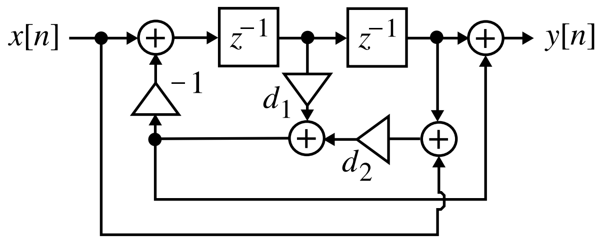
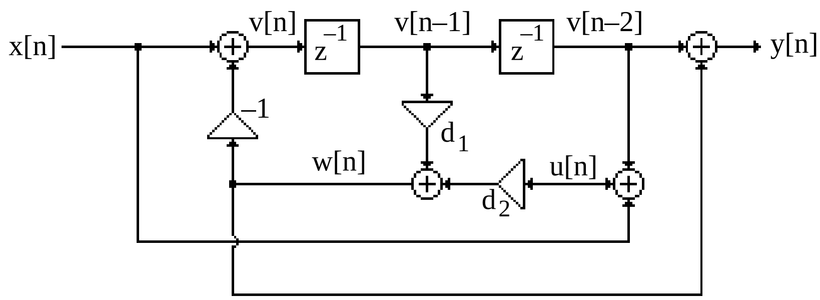


Figure E2.1

Answer: From the figure shown below we obtain



$v[n] = x[n] - w[n]$, $w[n] = d_1 v[n-1] + d_2 u[n]$, and $u[n] = v[n-2] + x[n]$. From these equations we get $w[n] = d_2 x[n] + d_1 x[n-1] + d_2 x[n-2] - d_1 w[n-1] - d_2 w[n-2]$. From the figure we also obtain $y[n] = v[n-2] + w[n] = x[n-2] + w[n] - w[n-2]$, which yields

$d_1 y[n-1] = d_1 x[n-3] + d_1 w[n-1] - d_1 w[n-3]$, and

$d_2 y[n-2] = d_2 x[n-4] + d_2 w[n-2] - d_2 w[n-4]$, Therefore,

$y[n] + d_1 y[n-1] + d_2 y[n-2] = x[n-2] + d_1 x[n-3] + d_2 x[n-4]$

$+\left(w[n] + d_1 w[n-1] + d_2 w[n-2]\right) - \left(w[n-2] + d_1 w[n-3] + d_2 w[n-4]\right)$

$= x[n-2] + d_2 x[n] + d_1 x[n-1]$ or equivalently,

$y[n] = d_2 x[n] + d_1 x[n-1] + x[n-2] - d_1 y[n-1] - d_2 y[n-2]$.

Example E2.2: The sequence $\{0 \quad -\sqrt{2} \quad -2 \quad -\sqrt{2} \quad 0 \quad \sqrt{2} \quad 2 \quad \sqrt{2}\}$ represents one period of a sinusoidal sequence $x[n] = A\sin(\omega_o n + \phi)$. Determine the values of the parameters A, ω_o, and ϕ.

Answer: Given $x[n] = \{0 \quad -\sqrt{2} \quad -2 \quad -\sqrt{2} \quad 0 \quad \sqrt{2} \quad 2 \quad \sqrt{2}\}$. The fundamental period is N = 4, hence $\omega_o = 2\pi / 8 = \pi / 4$. Next from $x[0] = A\sin(\phi) = 0$ we get $\phi = 0$, and solving $x[1] = A\sin(\frac{\pi}{4} + \phi) = A\sin(\pi / 4) = -\sqrt{2}$ we get A = –2.

Additional Examples of Chapter 2:
Discrete-Time Signals and Systems

Example E2.3: Determine the fundamental period of the periodic sequence $\tilde{x}[n] = \sin(0.6\pi n + 0.6\pi)$.

Answer: Here, $\omega_o = 0.6\pi$. From Eq. (2.47a), we thus get $N = \frac{2\pi r}{\omega_o} = \frac{2\pi r}{0.6\pi} = \frac{10}{3} r = 10$ for $r = 3$.

Example E2.4: Determine the fundamental period of the periodic sequence $\tilde{y}[n] = 3\sin(1.3\pi n) - 4\cos(0.3\pi n + 0.45\pi)$.

Answer: $N_1 = \frac{2\pi r_1}{1.3\pi} = \frac{20}{13} r_1$ and $N_2 = \frac{2\pi r_2}{0.3\pi} = \frac{20}{3} r_2$. To be periodic we must have $N_1 = N_2$. This implies, $\frac{20}{13} r_1 = \frac{20}{3} r_2$. This equality holds for $r_1 = 13$ and $r_2 = 7$, and hence N = $N_1 = N_2 = 20$.

Example E2.5: Let $\{y[n]\} = \{-1 \quad -1 \quad 11 \quad -3 \quad 30 \quad 28 \quad 48\}$ obtained by a linear convolution of the sequence $\{h[n]\} = \{-1 \quad 2 \quad 3 \quad 4\}$ with a finite-length sequence $\{x[n]\}$. The first sample in each sequence is time instant n = 0. Determine x[n].

Answer: The length of x[n] is 7 – 4 + 1 = 4. Using $x[n] = \frac{1}{h[0]}\left\{y[n] - \sum_{k=1}^{7} h[k]x[n-k]\right\}$ we arrive at x[n] = {1 3 –2 12}, $0 \le n \le 3$.

Example E2.6: Determine the expression for the impulse response of the LTI discrete-time system shown in Figure E2.2.

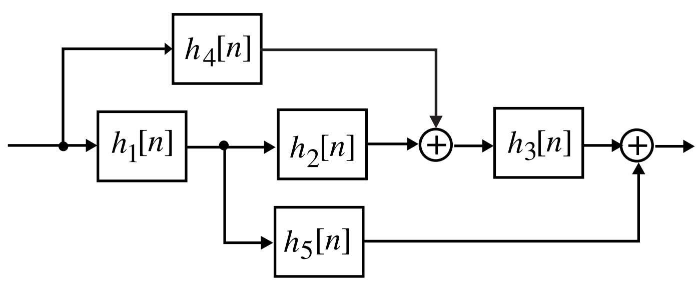
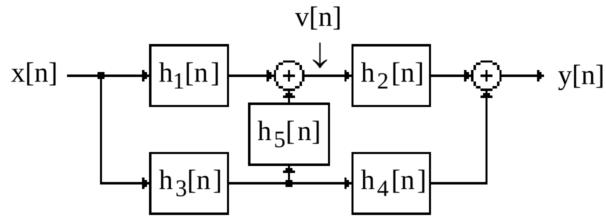


Figure E2.2

Answer: From the figure shown below we observe



$v[n] = (h_1[n] + h_3[n] \circledast h_5[n]) \circledast x[n]$ and $y[n] = h_2[n] \circledast v[n] + h_3[n] \circledast h_4[n] \circledast x[n]$.

Thus, $y[n] = (h_2[n] \circledast h_1[n] + h_2[n] \circledast h_3[n] \circledast h_5[n] + h_3[n] \circledast h_4[n]) \circledast x[n]$.

Hence the impulse response is given by

$$h[n] = h_2[n] \circledast h_1[n] + h_2[n] \circledast h_3[n] \circledast h_5[n] + h_3[n] \circledast h_4[n].$$

Example E2.7: Determine the total solution for $n \geq 0$ of the difference equation

$$y[n] + 0.1y[n-1] - 0.06y[n-2] = 2^n \mu[n],$$

with the initial condition $y[-1] = 1$ and $y[-2] = 0$.

Answer: $y[n] + 0.1y[n-1] - 0.06y[n-2] = 2^n \mu[n]$ with $y[-1] = 1$ and $y[-2] = 0$. The complementary solution $y_c[n]$ is obtained by solving $y_c[n] + 0.1y_c[n-1] - 0.06y_c[n-2] = 0$. To this end we set $y_c[n] = \lambda^n$, which yields $\lambda^n + 0.1\lambda^{n-1} - 0.06\lambda^{n-2} = \lambda^{n-2}(\lambda^2 + 0.1\lambda - 0.06) = 0$ whose solution gives $\lambda_1 = -0.3$ and $\lambda_2 = 0.2$. Thus, the complementary solution is of the form $y_c[n] = \alpha_1(-0.3)^n + \alpha_2(0.2)^n$.

For the particular solution we choose $y_p[n] = \beta(2)^n$. ubstituting this solution in the difference equation representation of the system we get $\beta 2^n + \beta(0.1)2^{n-1} - \beta(0.06)2^{n-2} = 2^n \mu[n]$. For $n = 0$ we get $\beta + \beta(0.1)2^{-1} - \beta(0.06)2^{-2} = 1$ or $\beta = 200/207 = 0.9662$.

The total solution is therefore given by $y[n] = y_c[n] + y_p[n] = \alpha_1(-0.3)^n + \alpha_2(0.2)^n + \frac{200}{207}2^n$.

From the above $y[-1] = \alpha_1(-0.3)^{-1} + \alpha_2(0.2)^{-1} + \frac{200}{207}2^{-1} = 1$ and $y[-2] = \alpha_1(-0.3)^{-2} + \alpha_2(0.2)^{-2} + \frac{200}{207}2^{-2} = 0$ or equivalently, $-\frac{10}{3}\alpha_1 + 5\alpha_2 = \frac{107}{207}$ and $\frac{100}{9}\alpha_1 + 25\alpha_2 = -\frac{50}{207}$ whose solution yields $\alpha_1 = -0.1017$ and $\alpha_2 = 0.0356$. Hence, the total solution is given by $y[n] = -0.1017(-0.3)^n + 0.0356(0.2)^n + 0.9662(2)^n$, for $n \geq 0$.

Example E2.8: Determine the total solution for $n \geq 0$ of the difference equation

$$y[n] + 0.1y[n-1] - 0.06y[n-2] = x[n] - 2x[n-1],$$

with the initial condition $y[-1] = 1$ and $y[-2] = 0$, when the forcing function is $x[n] = 2^n \mu[n]$.

Answer: $y[n]+0.1y[n-1]-0.06y[n-2]=x[n]-2x[n-1]$ with $x[n]=2^n\mu[n]$, and $y[-1]=1$ and $y[-2]=0$. For the given input, the difference equation reduces to $y[n]+0.1y[n-1]-0.06y[n-2]=2^n\mu[n]-2(2^{n-1})\mu[n-1]=\delta[n]$. The solution of this equation is thus the complementary solution with the constants determined from the given initial conditions y[–1] = 1 and y[–2] = 0.

From the solution of the previous problem we observe that the complementary solution is of the form $y_c[n]=\alpha_1(-0.3)^n+\alpha_2(0.2)^n$.

For the given initial conditions we thus have $y[-1]=\alpha_1(-0.3)^{-1}+\alpha_2(0.2)^{-1}=1$ and $y[-2]=\alpha_1(-0.3)^{-2}+\alpha_2(0.2)^{-2}=0$. Combining these two equations we get $\begin{bmatrix}-1/0.3 & 1/0.2\\ 1/0.09 & 1/0.04\end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}=\begin{bmatrix}1\\ 0\end{bmatrix}$ which yields $\alpha_1=-0.18$ and $\alpha_2=0.08$.

Therefore, $y[n]=-0.18(-0.3)^n+0.08(0.2)^n$.

Example E2.9: Determine the impulse response h[n] of the LTI discrete-time system described by the difference equation

$$y[n]+0.1y[n-1]-0.06y[n-2]=x[n]-2x[n-1].$$

Answer: The impulse response is given by the solution of the difference equation $y[n]+0.1y[n-1]-0.06y[n-2]=\delta[n]$. From Example E2.7, the complementary solution is given by $y_c[n]=\alpha_1(-0.3)^n+\alpha_2(0.2)^n$. To determine the constants α_1 and α_2, we observe y[0] = 1 and y[1] + 0.1y[0] = 0 as y[–1] = y[–2] = 0. From the complementary solution $y[0]=\alpha_1(-0.3)^0+\alpha_2(0.2)^0=\alpha_1+\alpha_2=1$, and $y[1]=\alpha_1(-0.3)^1+\alpha_2(0.2)^1=-0.3\alpha_1+0.2\alpha_2=-0.1$. Solution of these equations yields $\alpha_1=0.6$ and $\alpha_2=0.4$. Therefore, the impulse response is given by $h[n]=0.6(-0.3)^n+0.4(0.2)^n$.
