Additional Examples of Chapter 4: Digital Signal Processing of Continuous-Time Signals

Example E4.1: A continuous-time signal $x_a(t)$ is composed of a linear combination of sinusoidal signals of frequencies 250 Hz, 450 Hz, 1.0 kHz, 2.75 kHz, and 4.05 kHz. The signal $x_a(t)$ is sampled at a 1.5 kHz rate and the sampled sequence is passed through an ideal lowpass filter with a cutoff frequency of 750 Hz, generating a continuous-time signal $y_a(t)$. What are the frequency components present in $y_a(t)$?

Answer: Since the signal $x_a(t)$ is being sampled at 1.5 kHz rate, there will be multiple copies of the spectrum at frequencies given by $F_i \pm 1500$ k, where F_i is the frequency of the i-th sinusoidal component in $x_a(t)$. Hence,

$F_1 = 250$ Hz,	$F_{1m} = 250, 1250, 1750, \ldots,$ Hz
$F_2 = 450$ Hz,	$F_{2m} = 450, 1050, 1950 \ldots,$ Hz
$F_3 = 1000$ Hz,	$F_{3m} = 1000, 500, 2500 \ldots,$ Hz
$F_4 = 2750$ Hz,	$F_{4m} = 2750, 1250, 250, \ldots,$Hz
$F_5 = 4050$ Hz,	$F_{5m} = 34050, 1050, 450, \ldots,$Hz

So after filtering by a lowpass filter with a cutoff at 750 Hz, the frequencies of the sinusoidal components present in $y_a(t)$ are 250, 450, 500 Hz.

Example E4.2: A continuous-time signal $x_a(t)$ is composed of a linear combination of sinusoidal signals of frequencies F_1 Hz, F_2 Hz, F_3 Hz, and F_4 Hz. The signal $x_a(t)$ is sampled at a 8 kHz rate and the sampled sequence is passed through an ideal lowpass filter with a cutoff frequency of 3.8 kHz, generating a continuous-time signal $y_a(t)$ composed of three sinusoidal signals of frequencies 450 Hz, 625 Hz, and 950 Hz, respectively. What are the possible values of F_1, F_2, F_3, and F_4? Is your answer unique? If not, indicated another set of possible values for these frequencies.

Answer: One possible set of values are $F_1 = 450$ Hz, $F_2 = 625$ Hz, $F_3 = 950$ Hz and $F_4 = 7550$ Hz. Another possible set of values are $F_1 = 450$ Hz, $F_2 = 625$ Hz , $F_3 = 950$ Hz, $F_4 = 7375$ Hz. Hence the solution is not unique.

Example E4.3: The continuous-time signal

$$x_a(t) = 3\cos(400\pi t) + 5\sin(1200\pi t) + 6\cos(4400\pi t) + 2\sin(5200\pi t)$$

is sampled at a 4-kHz rate generating the sequence x[n]. Determine the exact expression of x[n].

Answer: $t = nT = \dfrac{n}{4000}$. Therefore,

$$x[n] = 3\cos\left(\frac{400\pi n}{4000}\right) + 5\sin\left(\frac{1200\pi n}{4000}\right) + 6\cos\left(\frac{4400\pi n}{4000}\right) + 2\sin\left(\frac{5200\pi n}{4000}\right)$$

$$= 3\cos\left(\frac{\pi n}{10}\right) + 5\sin\left(\frac{3\pi n}{10}\right) + 6\cos\left(\frac{11\pi n}{10}\right) + 2\sin\left(\frac{13\pi n}{10}\right)$$

$$= 3\cos\left(\frac{\pi n}{10}\right) + 5\sin\left(\frac{3\pi n}{10}\right) + 6\cos\left(\frac{(20-9)\pi n}{10}\right) + 2\sin\left(\frac{(20-7)\pi n}{10}\right)$$
$$= 3\cos\left(\frac{\pi n}{10}\right) + 5\sin\left(\frac{3\pi n}{10}\right) + 6\cos\left(\frac{9\pi n}{10}\right) - 2\sin\left(\frac{7\pi n}{10}\right).$$

Example E4.4: A continuous-time signal $x_a(t)$ has a band-limited spectrum $X_a(j\Omega)$ as indicated in Figure E4.1 with $\Omega_1 = 160\pi$ and $\Omega_2 = 200\pi$. Determine the smallest sampling rate F_T that can be employed to sample $x_a(t)$ so that it can be fully recovered from its sampled version x[n]. Sketch the Fourier transform $X_p(j\Omega)$ of x[n] and the frequency response of the ideal reconstruction filter needed to fully recover $x_a(t)$.

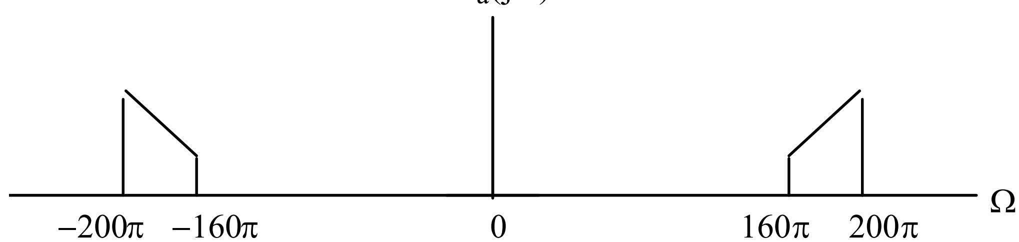
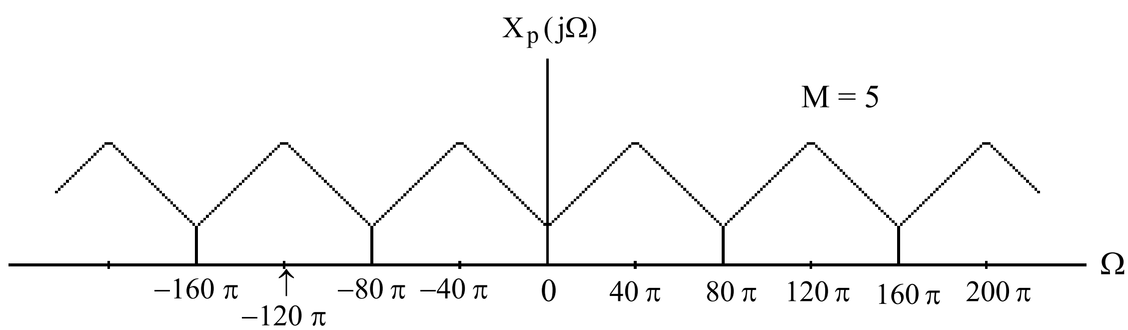


Figure E4.1

Answer: $\Omega_2 = 200\pi$ $\Omega_1 = 160\pi$ Thus, $\Delta\Omega = \Omega_2 - \Omega_1 = 40\pi$. Note $\Delta\Omega$ is an integer multiple of Ω_2. Hence, we choose the sampling angular frequency as $\Omega_T = 2\Delta\Omega = 2(\Omega_2 - \Omega_1) = 80\pi = \frac{2 \times 200\pi}{M}$, which is satisfied for M = 5. The sampling frequency is therefore $F_T = 40$ Hz.



Example E4.5: Determine the passband and stop ripples, δ_p and δ_s, of an analog lowpass filter with a peak passband deviation of $\alpha_p = 0.15$ dB and a minimum stopband attenuation of $\alpha_s = 43$ dB.

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Answer: $\alpha_p = -20\log_{10}(1-\delta_p)$ and $\alpha_s = -20\log_{10}\delta_s$. Therefore, $\delta_p = 1 - 10^{-\alpha_p/20}$ and $\delta_s = 10^{-\alpha_s/20}$. Hence, for $\alpha_p = 0.15$, $\alpha_s = 43.$, Hence, $\delta_p = 0.0171$ and $\delta_s = 0.0071$.

Example E4.6: Using Eq. (4.35) of text, determine the lowest order of an analog lowpass Butterworth filter with a 0.5 dB cutoff frequency at 2.1 kHz and a minimum attenuation of 30 dB at 8 kHz.

Answer: $10\log_{10}\left(\frac{1}{1+\varepsilon^2}\right) = -0.5,$, which yields $\varepsilon = 0.3493$. $10\log_{10}\left(\frac{1}{A^2}\right) = -30,$ which yields $A^2 = 1000$. Now, $\frac{1}{k} = \frac{\Omega_s}{\Omega_p} = \frac{8}{2.1} = 3.8095238$ and

$\frac{1}{k_1} = \frac{\sqrt{A^2-1}}{\varepsilon} = \frac{\sqrt{999}}{0.3493} = 90.486576.$ Then, from Eq. (4.35) we get

$N = \frac{\log_{10}(1/k_1)}{\log_{10}(1/k)} = \frac{90.4866}{3.8095} = 3.3684.$ Hence choose N = 4 as the order.

Example E4.7: Using Eq. (4.37) of text, determine the pole locations and the coefficients of a 5th-order Butterworth polynomial with a unity 3-dB cutoff frequency.

Answer: The poles are given by $p_\ell = \Omega_c e^{j\frac{\pi(N+2\ell-1)}{2N}}, \ell = 1, 2, \ldots, N.$ Here, N = 5 and $\Omega_c = 1$.
Hence $p_1 = e^{j6\pi/10} = -0.3090 + j0.9511$, $p_2 = e^{j8\pi/10} = -0.8090 + j\,0.5878$,
$p_3 = e^{j10\pi/10} = -1$, $p_4 = e^{j12\pi/10} = -0.8090 - j\,0.5878 = p_2^*$, and
$p_5 = e^{j6\pi/10} = -0.3090 - j0.9511 = p_1^*$.

The 5th-order Butterworth polynomial is thus given by
$B(s) = (s-p_1)(s-p_1^*)(s-p_2)(s-p_2^*)(s-p_3)$
$= (s+0.309+j0.9511)(s+0.309-j0.9511)(s+0.809+j0.5878)(s+0.809-j0.5878)(s+1)$
$= s^5 + 3.2361s^4 + 5.2361s^3 + 5.2361s^2 + 3.2361s + 1.$

Example E4.8: Using Eq. (4.43) of text, determine the lowest order of an analog lowpass Type 1 Chebyshev filter with a 0.5 dB cutoff frequency at 2.1 kHz and a minimum attenuation of 30 dB at 8 kHz.

Answer: $10\log_{10}\left(\frac{1}{1+\varepsilon^2}\right) = -0.5,$, which yields $\varepsilon = 0.3493$. $10\log_{10}\left(\frac{1}{A^2}\right) = -60,$ which yields $A^2 = 1000$. Now, $\frac{1}{k} = \frac{\Omega_s}{\Omega_p} = \frac{8}{2.1} = 3.8095238$ and

$\dfrac{1}{k_1} = \dfrac{\sqrt{A^2 - 1}}{\varepsilon} = \dfrac{\sqrt{999}}{0.3493} = 90.486576.$ Then, from Eq. (4.43) we get

$N = \dfrac{\cosh^{-1}(1/k_1)}{\cosh^{-1}(1/k)} = \dfrac{\cosh^{-1}(90.486576)}{\cosh^{-1}(3.8095238)} = \dfrac{5.1983}{2.013} = 2.5824.$ We thus choose N = 3 as the order.

Example E4.9: Using Eq. (4.54) of text, determine the lowest order of an analog lowpass elliptic filter with a 0.5 dB cutoff frequency at 2.1 kHz and a minimum attenuation of 30 dB at 8 kHz.

Answer: From Example E4.8, we observe $\dfrac{1}{k} = 3.8095238$ or $k = 0.2625$, and $\dfrac{1}{k_1} = 90.4865769$ or $k_1 = 0.011051362$. Substituting the value of k in Eq. (4.55a) we get k' = 0.964932. Then, from Eq. (4.55b) we arrive at $\rho_o = 0.017534$. Substituting the value of in Eq. (4.55c) we get ρ = 0.017534. Finally from Eq. (4.54) we arrive at N = 2.9139. We choose N = 3.

Example E4.10: The transfer function of a second-order analog Butterworth lowpass filter with a passband edge at 0.2 Hz and a passband ripple of 0.5 dB is given by

$$H_{LP}(s) = \frac{4.52}{s^2 + 3s + 4.52}.$$

Determine the transfer function $H_{HP}(s)$ of an analog highpass filter with a passband edge at 2 Hz and a passband ripple of 0.5 dB by applying the spectral transformation of Eq. (4.62).

Answer: We use the lowpass-to-highpass transformation given in Eq. (4.62) where now $\Omega_p = 2\pi(0.2) = 0.4\pi$ and $\hat{\Omega}_p = 2\pi(2) = 4\pi$. Therefore, from Eq. (4.62) we get the desired transformation as $s \rightarrow \dfrac{\Omega_p \hat{\Omega}_p}{s} = \dfrac{1.6\pi^2}{s} = \dfrac{15.791367}{s}$. Therefore,

$$H_{HP}(s) = H_{LP}(s)\Big|_{s \rightarrow \frac{15.791367}{s}} =$$

$$= \frac{4.52}{\left(\dfrac{15.791367}{s}\right)^2 + 3\left(\dfrac{15.791367}{s}\right) + 4.52} = \frac{4.52s^2}{4.52s^2 + 47.3741s + 2.49367}.$$

Example E4.11: The transfer function of a second-order analog elliptic lowpass filter with a passband edge at 0.16 Hz and a passband ripple of 1 dB is given by

$$H_{LP}(s) = \frac{0.056(s^2 + 17.95)}{s^2 + 1.06s + 1.13}.$$

Determine the transfer function $H_{BP}(s)$ of an analog highpass filter with a center frequency at 3 Hz and a 3-dB bandwidth of 0.5 Hz by applying the spectral transformation of Eq. (4.64).

Answer: We use the lowpass-to-bandpass transformation given in Eq. (4.64) where now $\Omega_p = 2\pi(0.16) = 0.32\pi$, $\hat{\Omega}_o = 2\pi(3) = 6\pi$, and $\hat{\Omega}_{p2} - \hat{\Omega}_{p1} = 2\pi(0.5) = \pi$. Substituting these values in Eq. (4.64) we get the desired transformation $s \rightarrow 0.32\pi\left(\frac{s^2 + 36\pi^2}{\pi s}\right) = \frac{s^2 + 36\pi^2}{3.125 s}$.

Therefore $H_{BP}(s) = H_{LP}(s)\Big|_{s \rightarrow \frac{s^2+36\pi^2}{3.125s}} = \dfrac{0.056\left[\left(\frac{s^2+36\pi^2}{3.125s}\right)^2 + 17.95\right]}{\left(\frac{s^2+36\pi^2}{3.125s}\right)^2 + 1.06\left(\frac{s^2+36\pi^2}{3.125s}\right) + 1.13}$

$$= \frac{0.056(s^4 + 49.61s^2 + 70695.62)}{s^4 + 3.3125s^3 + 721.64667s^2 + 1176.95s + 126242.182}.$$

Example E4.12: An analog Butterworth highpass filter is to be designed with the following specifications: $F_p = 5$ MHz, $F_s = 0.5$ MHz, dB, $\alpha_p = 0.3$ and $\alpha_s = 45$ dB. What are the bandedges and the order of the corresponding prototype analog lowpass filter? What is the order of the highpass filter? Verify your results using the function `buttord`.

Answer: $\hat{\Omega}_p = 2\pi(5)\times 10^6$, $\hat{\Omega}_s = 2\pi(0.5)\times 10^6$, $\hat{\alpha}_p = 0.3$ dB and $\hat{\alpha}_s = 45$ dB. From Figure 4.17 we observe $10\log_{10}\left(\frac{1}{1+\varepsilon^2}\right) = -0.3$, hence, $\varepsilon^2 = 10^{0.03} - 1 = 0.0715193$, or $\varepsilon = 0.26743$.

From Figure 4.17 we also note that $10\log_{10}\left(\frac{1}{A^2}\right) = -45$, and hence $A^2 = 10^{4.5} = 31622.7766$.

Using these values in Eq. (4.32) we get $k_1 = \frac{\varepsilon}{\sqrt{A^2 - 1}} = 0{,}001503898$.

To develop the bandedges of the lowpass prototype, we set $\Omega_p = 1$ and obtain using Eq. (4.32) $\Omega_s = \frac{\hat{\Omega}_p}{\hat{\Omega}_s} = \frac{5}{0.5} = 10$. Next, using Eq. (4.31) we obtain $k = \frac{\Omega_p}{\Omega_s} = \frac{1}{10} = 0.1$. Substituting the values of k and k_1 in Eq. (4.35) we get $N = \frac{\log_{10}(1/k_1)}{\log_{10}(1/k)} = 2.82278$. We choose N = 3 as the order of the lowpass filter which is also the order of the highpass filter.

To verify using MATLAB use the statement
`[N,Wn] = buttord(1,10,0.3,45,'s')` which yields `N = 3` and `Wn = 1.77828878`.

Example E4.13: An analog elliptic bandpass filter is to be designed with the following specifications: passband edges at 20 kHz and 45 kHz, stopband edges at 10 kHz and 60 kHz, peak passband ripple of 0.5 dB, and a minimum stopband attenuation of 40 dB. What are the bandedges and the order of the corresponding prototype analog lowpass filter? What is the order of the bandpass filter? Verify your results using the function `elliptord`.

Answer: $\hat{F}_{p1} = 20\times 10^3$, $\hat{F}_{p2} = 45\times 10^3$, $\hat{F}_{s1} = 10\times 10^3$, $\hat{F}_{s2} = 60\times 10^3$, $\alpha_p = 0.5$ dB and $\alpha_s = 40$ dB. We observe $\hat{F}_{p1}\hat{F}_{p2} = 20\times 45\times 10^6 = 9\times 10^8$, and $\hat{F}_{s1}\hat{F}_{s2} = 10\times 60\times 10^6 = 6\times 10^8$. Since $\hat{F}_{s1}\hat{F}_{s2} \neq \hat{F}_{p1}\hat{F}_{p2}$, we adjust the stopband edge on the left to $\hat{F}_{s1} = 15\times 10^3$ in which case $\hat{F}_{s1}\hat{F}_{s2} = \hat{F}_{p1}\hat{F}_{p2} = F_o^2 = 9\times 10^8$. The angular center frequency of the desired bandpass filter is therefore $\Omega_o = 2\pi F_o = 2\pi\times 30\times 10^3$. The passband bandwidth is
$B_w = \hat{\Omega}_{p2} - \hat{\Omega}_{p1} = 2\pi\times 25\times 10^3$.

To determine the bandedges of the prototype lowpass filter we set $\Omega_p = 1$ and then using Eq. (4.65) we obtain $\Omega_s = \Omega_p \dfrac{\hat{\Omega}_o^2 - \hat{\Omega}_{s1}^2}{\hat{\Omega}_{s1}B_w} = \dfrac{30^2 - 15^2}{15\times 25} = 1.8$. Thus, $k = \dfrac{\Omega_p}{\Omega_s} = \dfrac{1}{1.8} = 0.5555555556$.

Now, $10\log_{10}\left(\dfrac{1}{1+\varepsilon^2}\right) = -0.5$, and hence, $\varepsilon^2 = 10^{0.05} - 1 = 0.1220184543$, or

$\varepsilon = 0.34931140019$. In addition $10\log_{10}\left(\dfrac{1}{A^2}\right) = -40$ or $A^2 = 10^4 = 10000$. Using these values

in Eq. (4.32) we get $k_1 = \dfrac{\varepsilon}{\sqrt{A^2 - 1}} = 0.00349328867069$. From Eq. (4.55a) we get
$k' = \sqrt{1-k^2} = 0.831479419283$, and then from Eq. (4.55b) we get
$\rho_o = \dfrac{1-\sqrt{k'}}{2(1+\sqrt{k'})} = 0.02305223718137$. Substituting the value of ρ_o in Eq. (4.55c) we then get
$\rho = 0.02305225$. Finally, substituting the values of ρ and k_1 in Eq. (4.54) we arrive at
$N = \dfrac{2\log_{10}(4/k_1)}{\log_{10}(1/\rho)} = 3.7795$. We choose N = 4 as the order for the prototype lowpass filter.

The order of the desired bandpass filter is therefore 8.

Using the statement `[N,Wn]=ellipord(1,1.8,0.5,40,'s')` we get `N = 4` and `Wn = 1`.

Example E4.14: An analog elliptic bandstop filter is to be designed with the following specifications: passband edges at 10 MHz and 70 MHz, stopband edges at 20 MHz and 45 MHz, peak passband ripple of 0.5 dB, and a minimum stopband attenuation of 30 dB. What are the bandedges and the order of the corresponding prototype analog lowpass filter? What is the order of the bandstop filter? Verify your results using the function `cheb1ord`.

Answer: $\hat{F}_{p1} = 10 \times 10^6$, $\hat{F}_{p2} = 70 \times 10^6$, $\hat{F}_{s1} = 20 \times 10^6$, and $\hat{F}_{s2} = 45 \times 10^6$, We observe $\hat{F}_{p1}\hat{F}_{p2} = 700 \times 10^{12}$, and $\hat{F}_{s1}\hat{F}_{s2} = 900 \times 10^{12}$. Since $\hat{F}_{p1}\hat{F}_{p2} \neq \hat{F}_{s1}\hat{F}_{s2}$. we adjust the left passband edge to $\hat{F}_{p1} = (900 / 70) \times 10^6 = 12.8571 \times 10^6$, in which case $\hat{F}_{p1}\hat{F}_{p2} = \hat{F}_{s1}\hat{F}_{s2} = \hat{F}_o^2 = 700 \times 10^{12}$. The stopband bandwidth is $B_w = \hat{\Omega}_{s2} - \hat{\Omega}_{s1} = 2\pi \times 15 \times 10^6$.

Now, $10\log_{10}\left(\frac{1}{1+\varepsilon^2}\right) = -0.5$, and hence, $\varepsilon^2 = 10^{0.05} - 1 = 0.1220184543$, or $\varepsilon = 0.34931140019$. In addition $10\log_{10}\left(\frac{1}{A^2}\right) = -30$ or $A^2 = 10^3 = 1000$. Using these values in Eq. (4.32) we get $k_1 = \frac{\varepsilon}{\sqrt{A^2 - 1}} = 0.01105172361656$.

To determine the bandedges of the prototype lowpass filter we set $\Omega_s = 1$ and then using Eq. (4.68) we obtain $\Omega_p = \Omega_s \frac{\hat{\Omega}_{p1} B_w}{\hat{\Omega}_o^2 - \hat{\Omega}_{p1}^2} = 0.4375$. Therefore, $k = \frac{\Omega_p}{\Omega_s} = 0.4375$. Substituting the values of k and k_1 in Eq. (4.43) we get $N = \frac{\cosh^{-1}(1/k_1)}{\cosh^{-1}(1/k)} = 3.5408$. We thus choose N = 4 as the order for the prototype lowpass filter.

The order of the desired bandstop filter is therefore 8.
Using the statement `[N,Wn] = cheb1ord(0.4375,1,0.5,30,'s')` we get `N = 4` and `Wn = 0.4375`.
