Additional Examples of Chapter 9: IIR Digital Filter Design

Example E9.1: The peak passband ripple and the minimum stopband attenuation in dB of a digital filter are $\alpha_p = 0.15$ dB and $\alpha_s = 41$ dB. Determine the corresponding peak passband and stopband ripple values δ_p and δ_s.

Answer: $\delta_p = 1 - 10^{-\alpha_p/10}$ and $\delta_s = 10^{-\alpha_s/10}$. Hence $\delta_p = 1 - 10^{-0.15/20} = 0.017121127$ and $\delta_s = 10^{-41/20} = 0.0089125$.

Example E9.2: Determine the peak passband ripple α_p and the minimum stopband attenuation α_s in dB of a digital filter with peak passband ripple $\delta_p = 0.035$ and peak stopband ripple $\delta_s = 0.023$.

Answer: $\alpha_p = -20\log_{10}(1-\delta_p)$ and $\alpha_s = -20\log_{10}(\delta_s)$. Hence,
$\alpha_p = -20\log_{10}(1-0.035) = 0.3094537$ dB and $\alpha_s = -20\log_{10}(0.023) = 32.76544$ dB.

Example E9.3: Determine the digital transfer function obtained by transforming the causal analog transfer function

$$H_a(s) = \frac{16(s+2)}{(s+3)(s^2+2s+5)}$$

using the impulse invariance method. Assume T = 0.2 sec.

Answer: Applying partial-fraction expansion we can express

$$H_a(s) = 16\left[\frac{-1/8}{s+3} + \frac{0.0625 - j0.1875}{s+1-j2} + \frac{0.0625 + j0.1875}{s+1+j2}\right]$$

$$= 16\left[\frac{-\frac{1}{8}}{s+3} + \frac{\frac{1}{8}s + \frac{7}{8}}{s^2+2s+5}\right] = \frac{-2}{s+3} + \frac{2s+14}{(s+1)^2+2^2} = \frac{-2}{s+3} + \frac{2(s+1)}{(s+1)^2+2^2}\frac{6\times 2}{(s+1)^2+2^2}.$$

Using the results of Problems 9.7, 9.8 and 9.9 we thus arrive at

$$G(z) = -\frac{2}{1-e^{-3T}z^{-1}} + \frac{2\left(z^2 - ze^{-2T}\cos(2T)\right)}{z^2 - 2ze^{-2T}\cos(2T) + e^{-4T}} + \frac{6ze^{-2T}\sin(2T)}{z^2 - 2ze^{-2T}\cos(2T) + e^{-4T}}.$$ For T = 0.2, we then get

$$G(z) = -\frac{2}{1-e^{-0.6}z^{-1}} + \frac{2\left(z^2 - ze^{-0.4}\cos(0.4)\right)}{z^2 - 2ze^{-0.4}\cos(0.4) + e^{-0.8}} + \frac{6ze^{-0.4T}\sin(0.4)}{z^2 - 2ze^{-0.4}\cos(0.4) + e^{-0.8}}$$

$$= -\frac{2}{1-0.5488z^{-1}} + \frac{2\left(z^2 - 0.6174z\right)}{z^2 - 1.2348z + 0.4493} + \frac{1.5662z}{z^2 - 1.2348z + 0.4493}$$

$$= -\frac{2}{1-0.5488z^{-1}} + \frac{2-1.2348z^{-1}}{1-1.2348z^{-1}+0.4493z^{-2}} + \frac{1.5662z^{-1}}{1-1.2348z^{-1}+0.4493z^{-2}}$$

$$= -\frac{2}{1-0.5488z^{-1}} + \frac{2+0.3314z^{-1}}{1-1.2348z^{-1}+0.4493z^{-2}}.$$

Example E9.4: The causal digital transfer function

$$G(z) = \frac{2z}{z-e^{-0.9}} + \frac{3z}{z-e^{-1.2}}$$

was designed using the impulse invariance method with T = 2. Determine the parent analog transfer function.

Answer: Comparing G(z) with Eq. (9.59) we can write $G(z) = \frac{2}{1-e^{-0.9}z^{-1}} + \frac{3}{1-e^{-1.2}z^{-1}} = \frac{2}{1-e^{-\alpha T}z^{-1}} + \frac{3}{1-e^{-\beta T}z^{-1}}$. Hence, $\alpha = 3$ and $\beta = 4$.

Therefore, $H_a(s) = \frac{2}{s+3} + \frac{3}{s+4}$.

Example E9.5: The causal IIR digital transfer function

$$G(z) = \frac{5z^2+4z-1}{8z^2+4z}$$

was designed using the bilinear transformation method with T = 2. Determine the parent analog transfer function.

Answer: $H_a(s) = G(z)\Big|_{z=\frac{1+s}{1-s}} = \dfrac{5\left(\frac{1+s}{1-s}\right)^2 + 4\left(\frac{1+s}{1-s}\right) - 1}{8\left(\frac{1+s}{1-s}\right)^2 + 4\left(\frac{1+s}{1-s}\right)} = \dfrac{2+3s}{s^2+4s+3}$.

Example E9.6: A lowpass IIR digital transfer function is to be designed by transforming a lowpass analog filter with a passband edge F_p at 0.5 kHz using the impulse invariance method with T = 0.5 ms. What is the normalized passband edge angular frequency ω_p of the digital filter if the effect of aliasing is negligible? What is the normalized passband edge angular frequency ω_p of the digital filter if it is designed using the bilinear transformation method with T = 0.5 ms?

Answer: For the impulse invariance design $\omega_p = \Omega_p T = 2\pi \times 0.5 \times 10^3 \times 0.5 \times 10^{-3} = 0.5\pi$.

For the bilinear transformation method design $\omega_p = 2\tan^{-1}\left(\frac{\Omega_p T}{2}\right)$

$= 2\tan^{-1}(\pi F_p T) = 2\tan^{-1}(0.25\pi) = 0.4238447331\pi$.

Additional Examples of Chapter 9: IIR Digital Filter Design

Example E9.7: A lowpass IIR digital filter has a normalized passband edge at $\omega_p = 0.3\pi$. What is the passband edge frequency in Hz of the prototype analog lowpass filter if the digital filter has been designed using the impulse invariance method with T = 0.1 ms? What is the passband edge frequency in Hz of the prototype analog lowpass filter if the digital filter has been designed using the bilinear transformation method with T = 0.1 ms?

Answer: For the impulse invariance design $2\pi F_p = \dfrac{\omega_p}{T} = \dfrac{0.3\pi}{10^{-4}}$ or $F_p = 1.5$ kHz. For the bilinear transformation method design $F_p = 10^4 \tan(0.15\pi) / \pi = 1.62186$ kHz.

Example E9.8: The transfer function of a second-order lowpass IIR digital filter with a 3-dB cutoff frequency at $\omega_c = 0.42\pi$ is

$$G_{LP}(z) = \frac{0.223(1+z^{-1})^2}{1-0.2952\,z^{-1}+0.187z^{-2}}.$$

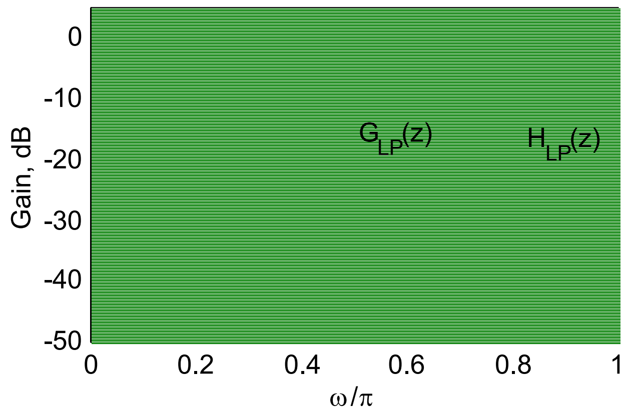
Design a second-order lowpass filter $H_{LP}(z)$ with a 3-dB cutoff frequency at $\hat{\omega}_c = 0.57\pi$ by transforming $G_{LP}(z)$ using a lowpass-to-lowpass spectral transformation. Using MATLAB plot the gain responses of the two lowpass filters on the same figure.

Answer: For $\omega_c = 0.42\pi$ and $\hat{\omega}_c = 0.57\pi$ we have

$$\alpha = \frac{\sin\left(\dfrac{\omega_c - \hat{\omega}_c}{2}\right)}{\sin\left(\dfrac{\omega_c + \hat{\omega}_c}{2}\right)} = \frac{\sin(-0.075\pi)}{\sin(0.495\pi)} = -0.233474.$$

Thus, $$H_{LP}(\hat{z}) = G_{LP}(z)\Big|_{z^{-1} = \frac{\hat{z}^{-1}-\alpha}{1-\alpha\hat{z}^{-1}}} = \frac{0.223\left(1+\dfrac{\hat{z}^{-1}-\alpha}{1-\alpha\hat{z}^{-1}}\right)^2}{1-0.2952\left(\dfrac{\hat{z}^{-1}-\alpha}{1-\alpha\hat{z}^{-1}}\right)+0.187\left(\dfrac{\hat{z}^{-1}-\alpha}{1-\alpha\hat{z}^{-1}}\right)^2}$$

$$= \frac{0.360454(1+\hat{z}^{-1})^2}{1+258136\hat{z}^{-1}+0.1833568\hat{z}^{-2}}.$$



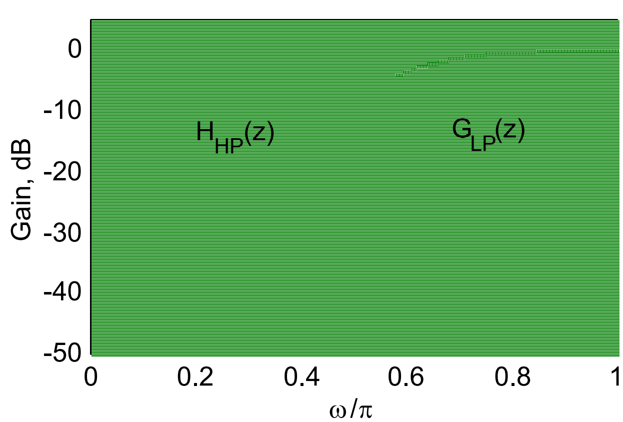
Example E9.9: Design a second-order highpass filter $H_{HP}(z)$ with a 3-dB cutoff frequency at $\hat{\omega}_c = 0.61\pi$ by transforming $G_{LP}(z)$ of Example E9.8 using the lowpass-to-highpass spectral transformation. Using MATLAB plot the gain responses of the both filters on the same figure.

Answer: For $\omega_c = 0.42\pi$ and $\hat{\omega}_c = 0.61\pi$ we have

$$\alpha = -\frac{\cos\left(\frac{\omega_c + \hat{\omega}_c}{2}\right)}{\cos\left(\frac{\omega_c - \hat{\omega}_c}{2}\right)} = -\frac{\cos(0.515\pi)}{\cos(-0.95\pi)} = 0.0492852.$$

$$H_{HP}(\hat{z}) = G_{LP}(z)\Big|_{z^{-1} = -\frac{\hat{z}^{-1}+\alpha}{1+\alpha\hat{z}^{-1}}} = \frac{0.223\left(1 - \frac{\hat{z}^{-1}+\alpha}{1+\alpha\hat{z}^{-1}}\right)^2}{1 + 0.2952\left(\frac{\hat{z}^{-1}+\alpha}{1+\alpha\hat{z}^{-1}}\right) + 0.187\left(\frac{\hat{z}^{-1}+\alpha}{1+\alpha\hat{z}^{-1}}\right)^2}$$

$$= \frac{0.19858(1-\hat{z}^{-1})^2}{1 + 0.4068165\hat{z}^{-1} + 0.200963\hat{z}^{-2}}.$$



Additional Examples of Chapter 9: IIR Digital Filter Design

Example E9.10: The transfer function of a second-order lowpass Type 1 Chebyshev IIR digital filter with a 0.5-dB cutoff frequency at $\omega_c = 0.27\pi$ is

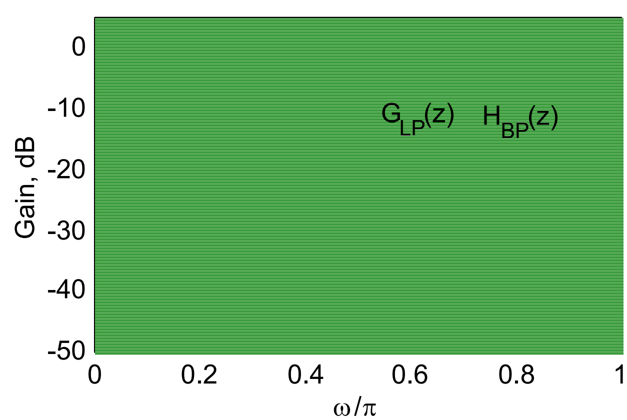
$$G_{LP}(z) = \frac{0.1494(1+z^{-1})^2}{1-0.7076\,z^{-1}+0.3407\,z^{-2}}.$$

Design a fourth-order bandpass filter $H_{BP}(z)$ with a center frequency at $\hat{\omega}_o = 0.45\pi$ by transforming $G_{LP}(z)$ using the lowpass-to-bandpass spectral transformation. Using MATLAB plot the gain responses of the both filters on the same figure.

Answer: Since the passband edge frequencies are not specified, we use the mapping of Eq. (9.44) to map $\omega = 0$ point of the lowpass filter $G_{LP}(z)$ to the specified center frequency $\hat{\omega}_o = 0.45\pi$ of the desired bandpass filter $H_{BP}(z)$. From Eq. (9.46) we get $\lambda = \cos(\hat{\omega}_o) = 0.1564347$. Substituting this value of in Eq. (9.44) we get the desired lowpass-to-bandpass transformation as $z^{-1} \to -z^{-1}\frac{z^{-1}-\lambda}{1-\lambda z^{-1}} = \frac{0.1564347z^{-1}-z^{-2}}{1-0.1564347\,z^{-1}}$.

Then, $H_{BP}(z) = G_{LP}(z)\Big|_{z^{-1}\to\frac{0.1564347z^{-1}-z^{-2}}{1-0.1564347z^{-1}}}$

$$= \frac{0.1494(1-z^{-2})^2}{1-0.423562z^{-1}+0.757725z^{-2}-0.217287z^{-3}+0.3407\,z^{-4}}.$$



Example E9.11: A third-order Type 1 Chebyshev highpass filter with a passband edge at $\omega_p = 0.6\pi$ has a transfer function

$$G_{HP}(z) = \frac{0.0916(1-3z^{-1}+3z^{-2}-z^{3})}{1+0.7601\,z^{-1}+0.7021\,z^{-2}+0.2088z^{3}}.$$

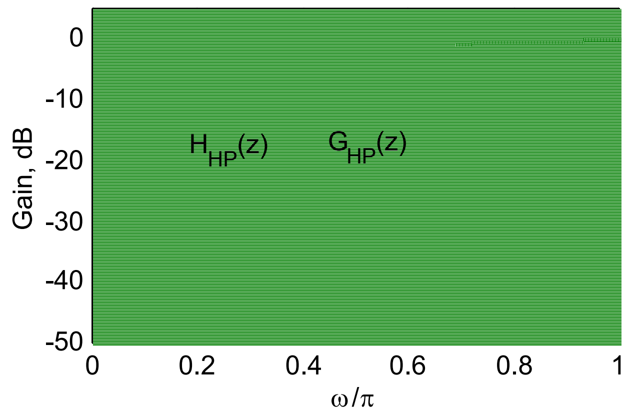
Design a highpass filter with a passband edge at $\hat{\omega}_p = 0.5\pi$ by transforming using the lowpass-to-lowpass spectral transformation. Using MATLAB plot the gain responses of the both filters on the same figure.

Additional Examples of Chapter 9:
IIR Digital Filter Design

Answer: $\omega_p = 0.6\pi$, and $\hat{\omega}_p = 0.5\pi$, Thus, $\alpha = \dfrac{\sin\left(\dfrac{\omega_p - \hat{\omega}_p}{2}\right)}{\sin\left(\dfrac{\omega_p + \hat{\omega}_p}{2}\right)} = \dfrac{\sin(0.05\pi)}{\sin(0.55\pi)} = 0.15838444.$

Therefore, $H_{HP}(z) = G_{HP}(z)\Big|_{z^{-1} \to \frac{z^{-1} - 0.15838444}{1 - 0.15838444\, z^{-1}}}$

$$= \frac{0.15883792(1 - z^{-1})^3}{1 + 0.126733 z^{-1} + 0.523847 z^{-2} + 0.125712 z^{3}}$$



Example E9.12: The transfer function of a second-order notch filter with a notch frequency at 60 Hz and operating at a sampling rate of 400 Hz is

$$G_{BS}(z) = \frac{0.954965 - 1.1226287 z^{-1} + 0.954965 z^{-2}}{1 - 1.1226287 z^{-1} + 0.90993 z^{-2}}$$

Design a second-order notch filter $H_{BS}(z)$ with a notch frequency at 100 Hz by transforming $G_{BS}(z)$ using the lowpass-to-lowpass spectral transformation. Using MATLAB plot the gain responses of the both filters on the same figure.

Answer: The above notch filter has notch frequency at $\omega_o = 2\pi\left(\dfrac{60}{400}\right) = 0.3\pi$. The desired notch frequency of the transformed filter is $\hat{\omega}_o = 2\pi\left(\dfrac{100}{400}\right) = 0.5\pi$. The lowpass-to-lowpass transformation to be used is thus given by $z^{-1} \to \dfrac{z^{-1} - \lambda}{1 - \lambda z^{-1}}$ where $\lambda = \dfrac{\sin\left(\dfrac{\omega_o - \hat{\omega}_o}{2}\right)}{\sin\left(\dfrac{\omega_o + \hat{\omega}_o}{2}\right)} =$ – 0.32492. The desired transfer function is thus given by $H_{BS}(z) = G_{BS}(z)\Big|_{z^{-1} = \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}}}$

$$= \frac{0.954965 - 1.1226287\left(\dfrac{z^{-1} - \lambda}{1 - \lambda z^{-1}}\right) + 0.954965\left(\dfrac{z^{-1} - \lambda}{1 - \lambda z^{-1}}\right)^2}{1 - 1.1226287\left(\dfrac{z^{-1} - \lambda}{1 - \lambda z^{-1}}\right) + 0.90993\left(\dfrac{z^{-1} - \lambda}{1 - \lambda z^{-1}}\right)^2}$$

$$= \frac{0.9449 - 0.1979 \times 10^{-7} z^{-1} + 0.9449\, z^{-2}}{1 - 0.1979 \times 10^{-7} z^{-1} + 0.8898 z^{-2}}.$$