Additional Examples of Chapter 6: Discrete-Time Signals and Systems in the *z*-Domain

Example E6.1: Consider the z-transform

$$H(z) = \frac{(z+0.4)(z^2+0.3z+0.4)}{(z-0.5)(z^2+0.5z+0.64)}.$$

There are 3 possible non-overlapping ROCs of H(z). Discuss the type of inverse z-transform (left-sided, right-sided, or two-sided sequences) associated with each of the three ROCs. Do not compute the exact inverse z-transform for each ROC.

Answer: The poles of H(z) are at $p_1 = 0.5$ and $p_{2,3} = -0.2500 \pm j0.7599$. Hence, the three ROCs are as follows: $\Re_1: 0 \le |z| < 0.5$, $\Re_2: 0.5 < |z| < 0.8$, and $\Re_3: 0.8 < |z| < \infty$. The sequence associated with the ROC $\Re_1$ is a left-sided sequence, the sequence associated with the ROC $\Re_2$ is a two-sided sequence, and the sequence associated with the ROC $\Re_3$ is a right-sided sequence.

Example E6.2: Determine the z-transform X(z) of $x[n] = a^n \mu[n+2]$, $|a| < 1$. Show that the ROC of X(z) includes the unit circle and determine the DTFT of x[n] by evaluating X(z) on the unit circle.

Answer: Therefore, $X(z) = \sum_{n=-\infty}^{\infty} a^n \mu[n+1] z^{-n} = \sum_{n=-1}^{\infty} a^n z^{-n} = \frac{z}{a}\sum_{n=0}^{\infty} a^n z^{-n}$ $= \frac{1}{az^{-1}(1-az^{-1})}, |z| > |a|$. The ROC of X(z) includes the unit circle since $|a| < 1$. On the unit circle $X(e^{j\omega}) = X(z)\big|_{z=e^{j\omega}} = \frac{1}{ae^{-j\omega}(1-ae^{-j\omega})}$, which is the DTFT of x[n].

Example E6.3: Determine the inverse z-transform x[n] of $X(z) = (1-z^{-2})^{-1}$, $|z| > 1$.

Answer: Using partial fraction, we get $X(z) = \frac{1}{1-z^{-2}} = \frac{\frac{1}{2}}{1+z^{-1}} + \frac{\frac{1}{2}}{1-z^{-1}}$. Therefore,

$$x[n] = \tfrac{1}{2}\mu[n] + \tfrac{1}{2}(-1)^n \mu[n] = \begin{cases} 1, & \text{if } n = 2k \text{ and } n \ge 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Example E6.4: Let X(z) denote the z-transform of the sequence $x[n] = \begin{cases} (0.4)^n, & n \ge 0, \\ 0, & n < 0. \end{cases}$ (a) Determine the inverse z-transform of $X(z^2)$ without computing X(z).

(b) Determine the inverse z-transform of $(1+z^{-1})X(z^2)$ without computing X(z).

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Answer:

Example E6.5: Determine the linear convolution of $\{g[n]\} = \{-2, \quad 1, \quad -3, \quad 4\}$ and $\{h[n]\} = \{1, \quad 2, \quad -3, \quad 2\}$, $0 \le n \le 3$, using the polynomial multiplication method.

Answer: The z-transforms of g[n] and h[n] are, respectively, given by $G(z) = -2 + z^{-1} - 3z^{-2} + 4z^{-3}$ and $H(z) = 1 + 2z^{-1} - 3z^{-2} + 2z^{-3}$. Hence
$G(z)H(z) = (-2 + z^{-1} - 3z^{-2} + 4z^{-3})(1 + 2z^{-1} - 3z^{-2} + 2z^{-3})$
$= -2 - 3z^{-1} + 5z^{-2} - 9z^{-3} + 19z^{-4} - 18z^{-5} + 8z^{-6}$. The sequence obtained by a linear convolution is therefore a length-7 sequence given by $\{-2, \quad -3, \quad 5, \quad -9, \quad 19, \quad -18, \quad 8\}$.

Example E6.6: Determine the circular convolution of $\{g[n]\} = \{-2, \quad 1, \quad -3, \quad 4\}$ and $\{h[n]\} = \{1, \quad 2, \quad -3, \quad 2\}$, $0 \le n \le 3$, using the polynomial multiplication method.

Answer: From Example E6.6 we have
$G(z)H(z) = -2 - 3z^{-1} + 5z^{-2} - 9z^{-3} + 19z^{-4} - 18z^{-5} + 8z^{-6}$. The length-4 sequence obtained by a circular convolution is thus given by
$\langle G(z)H(z)\rangle_{(z^{-4}-1)}) = \langle -2 - 3z^{-1} + 5z^{-2} - 9z^{-3} + 19z^{-4} - 18z^{-5} + 8z^{-6}\rangle_{(z^{-4}-1)}$
$= -2 - 3z^{-1} + 5z^{-2} - 9z^{-3} + 19 - 18z^{-1} + 8z^{-2} = = 17 - 21z^{-1} + 13z^{-2} - 9z^{-3}$.

The sequence obtained by a linear convolution is therefore a length-4 sequence given by $\{17, \quad -21, \quad 13, \quad -9\}$

Example E6.7: Consider the digital filter structure of Figure E6.1, where $H_1(z)$, $H_2(z)$, and $H_3(z)$ are FIR digital filters with transfer functions given by $H_1(z) = \frac{2}{3} + \frac{2}{5}z^{-1} + \frac{4}{7}z^{-2}$, $H_2(z) = \frac{4}{3} + \frac{8}{5}z^{-1} + \frac{3}{7}z^{-2}$, and $H_3(z) = 3 + 2z^{-1} + 4z^{-2}$. Determine the transfer function H(z) of the composite filter.

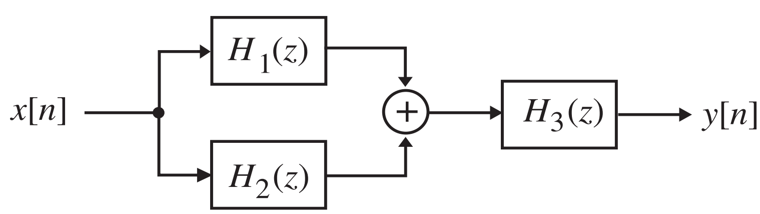

Figure E6.1

Answer: $H(z) = H_1(z)\left[H_2(z) + H_3(z)\right] = (3 + 2z^{-1} + 4z^{-2})\left(\left(\frac{2}{3} + \frac{4}{3}\right) + \left(\frac{2}{5} + \frac{8}{5}\right)z^{-1} + \left(\frac{4}{7} + \frac{3}{7}\right)z^{-2}\right)$
$= (3 + 2z^{-1} + 4z^{-2})(2 + 2z^{-1} + z^{-2}) = 6 + 10z^{-1} + 15z^{-2} + 10z^{-3} + 4z^{-4}$.

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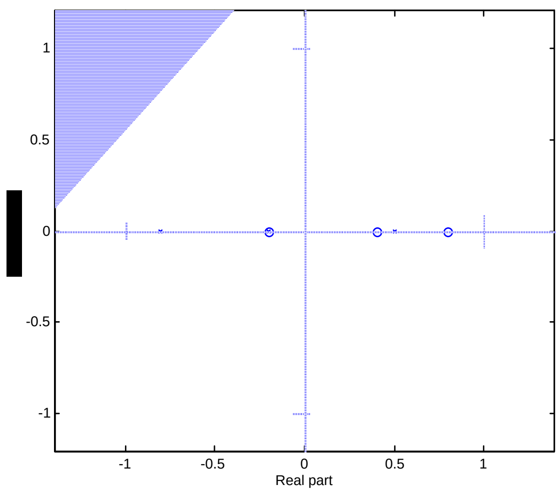
Example E6.8: Determine the transfer function of a causal LTI discrete-time system described by the difference equation

$$y[n] = 5x[n] - 5x[n-1] + 0.4x[n-2] + 0.32x[n-3] - 0.5y[n-1] + 0.34y[n-2] + 0.08y[n-3]$$

Express the transfer function in a factored form and sketch its pole-zero plot. Is the system BIBO stable?

Answer: $H(z) = \dfrac{5 - 5z^{-1} + 0.4z^{-2} + 0.32z^{-3}}{1 + 0.5z^{-1} - 0.34z^{-2} - 0.08z^{-3}} = 5 \cdot \dfrac{(1-0.8z^{-1})(1-0.4z^{-1})(1+0.2z^{-1})}{(1+0.8z^{-1})(1-0.5z^{-1})(1+0.2z^{-1})}$

$= 5 \cdot \dfrac{(1-0.8z^{-1})(1-0.4z^{-1})}{(1+0.8z^{-1})(1-0.5z^{-1})}$, obtained using the M-file `roots`. The pole-zero plot obtained using the M-file `zplane` is shown below. It can be seen from this plot and the factored form of H(z), the transfer function is BIBO stable as all poles are inside the unit circle. Note also from the plot the pole-zero cancellation at $z = -0.2$.



Example E6.9: Determine the expression for the impulse response h[n] of the following causal IIR transfer function:

$$H(z) = \frac{-0.1z^{-1} + 2.19z^{-2}}{(1-0.8z^{-1}+0.41z^{-2})(1+0.3z^{-1})}.$$

Answer: $H(z) = \dfrac{-0.1z^{-1} + 2.19z^{-2}}{(1-0.8z^{-1}+0.4z^{-2})(1+0.3z^{-1})} = \dfrac{A + Bz^{-1}}{1-0.8z^{-1}+0.4z^{-2}} + \dfrac{C}{1+0.3z^{-1}}$, where

$C = \left.\dfrac{-0.1z^{-1} + 2.19z^{-2}}{1-0.8z^{-1}+0.4z^{-2}}\right|_{z^{-1}=-1/0.3} = 3$, Thus,

$$\frac{A + Bz^{-1}}{1-0.8z^{-1}+0.4z^{-2}} = \frac{-0.1z^{-1} + 2.19z^{-2}}{(1-0.8z^{-1}+0.4z^{-2})(1+0.3z^{-1})} - \frac{3}{1+0.3z^{-1}} = \frac{-3+3.2z^{-1}}{1-0.8z^{-1}+0.4z^{-2}}$$

Hence, $H(z) = \frac{-3+3.2z^{-1}}{1-0.8z^{-1}+0.4z^{-2}} + \frac{3}{1+0.3z^{-1}} = -3\left[\frac{1-1.0667z^{-1}}{1-0.8z^{-1}+0.4z^{-2}}\right] + \frac{3}{1+0.3z^{-1}}$. Now, using Table 6.1 we observe $r^2 = 0.4$ and $r\cos\omega_o = 0.8$. Therefore, $r = \sqrt{0.4} = 0.6325$ and $\cos\omega_o = \sqrt{0.4}$ or $\omega_o = \cos^{-1}\left(\sqrt{0.4}\right) = 0.8861$. Hence $r\sin\omega_o = \sqrt{0.4}\sin(0.8861) = 0.4899$. We can thus write $\frac{1-1.0667z^{-1}}{1-0.8z^{-1}+0.4z^{-2}} = \frac{1-0.4z^{-1}}{1-0.8z^{-1}+0.4z^{-2}} - 1.3609\left(\frac{0.4899z^{-1}}{1-0.8z^{-1}+0.4z^{-2}}\right)$. The inverse z-transform of this function is thus given by
$(0.6325)^n\cos(0.8861n)\mu[n] - 1.3609(0.6325)^n\sin(0.8861n)\mu[n]$. Hence, the inverse z-transform of H(z) is given by

$$h[n] = 3(0.6325)^n\cos(0.8861n)\mu[n] - 4.0827(0.6325)^n\sin(0.8861n)\mu[n] - 3(-0.3)^n\mu[n].$$

Example E6.10: The transfer function of a causal LTI discrete-time system is given by

$$H(z) = \frac{6-z^{-1}}{1+0.5z^{-1}} + \frac{2}{1-0.4z^{-1}}.$$

(a) Determine the impulse response h[n] of the above system.

(b) Determine the output y[n] of the above system for all values of n for an input

$$x[n] = 1.2(-0.2)^n\mu[n] - 0.2(0.3)^n\mu[n].$$

Answer: (a) $H(z) = \frac{6-z^{-1}}{1+0.5z^{-1}} + \frac{2}{1-0.4z^{-1}} = k + \frac{A}{1+0.5z^{-1}} + \frac{2}{1-0.4z^{-1}}$ where

$A = (6-z^{-1})\Big|_{z^{-1}=-2} = 8$ and $k = \left.\frac{6-z^{-1}}{1+0.5z^{-1}}\right|_{z^{-1}=\infty} = -2$. Therefore,

$H(z) = -2 + \frac{8}{1+0.5z^{-1}} + \frac{2}{1-0.4z^{-1}}$. Hence, the inverse z-transform of H(z) is given by $h[n] = -2\delta[n] + 8(-0.5)^n\mu[n] + 2(0.4)^n\mu[n]$.

(b) The z-transform of x[n] is given by

$$X(z) = \frac{1.2}{1+0.2z^{-1}} - \frac{0.2}{1-0.3z^{-1}} = \frac{1-0.4z^{-1}}{(1+0.2z^{-1})(1-0.3z^{-1})}, \quad |z| > 0.3.$$ Therefore,

$$Y(z) = H(z)X(z) = \left[\frac{6-z^{-1}}{1+0.5z^{-1}} + \frac{2}{1-0.4z^{-1}}\right]\frac{1-0.4z^{-1}}{(1+0.2z^{-1})(1-0.3z^{-1})}$$

$$= \left[\frac{8-2.4z^{-1}+0.4z^{-2}}{(1+0.5z^{-1})(1-0.4z^{-1})}\right]\frac{1-0.4z^{-1}}{(1+0.2z^{-1})(1-0.3z^{-1})} = \frac{8-2.4z^{-1}+0.4z^{-2}}{(1+0.5z^{-1})(1+0.2z^{-1})(1-0.3z^{-1})},$$

$|z| > 0.5$.

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A partial-fraction expansion of Y(z) yields $Y(z) = \frac{15}{1+0.5z^{-1}} - \frac{8}{1+0.2\,z^{-1}} + \frac{1}{1-0.3z^{-1}}$. Hence, the inverse z-transform of Y(z) is given by $y[n] = 15(-0.5)^n \mu[n] - 8(-0.2)^n \mu[n] + (0.3)^n \mu[n]$.

Example E6.11: Determine the magnitude and phase responses of the causal IIR transfer function

$$H(z) = \frac{\alpha z^{-1}}{1-\alpha z^{-1}}.$$

Answer: The frequency response of the above transfer function is given by

$$H(e^{j\omega}) = \frac{\alpha e^{-j\omega}}{1-\alpha e^{-j\omega}} = \frac{\alpha e^{-j\omega}}{(1-\alpha\cos\omega) + j\alpha\sin\omega}.$$

Hence the magnitude responses is given by

$$\left|H(e^{j\omega})\right| = \frac{\alpha}{\sqrt{(1-\alpha\cos\omega)^2 + \alpha^2\sin^{2\omega}}} = \frac{\alpha}{\sqrt{2(1-\alpha\cos\omega)}}$$

and the phase response is given by

$$\arg\left[H(e^{j\omega})\right] = -\omega - \tan^{-1}\left(\frac{\alpha\sin\omega}{1-\alpha\cos\omega}\right).$$
