Additional Examples of Chapter 12: Analysis of Finite Wordlength Effects

Example E12.1: A third-order elliptic highpass transfer function

$$H(z) = \frac{0.1868(z-1)(z^2-0.0902z+1)}{(z+0.3628)(z^2+0.5111z+0.7363)}$$

is realized in (1) direct form, and (2) cascade form. Compute the pole sensitivities of each structure.

Answer: (a) For direct form implementation
$B(z) = z^3 + b_2 z^2 + b_1 z + b_0 = (z-z_1)(z-z_2)(z-z_3)$ where $z_1 = r_1 e^{j\theta_1}$, $z_2 = r_1 e^{-j\theta_1}$, and $z_3 = r_3 e^{j\theta_3}$. Thus, $B(z) = (z^2 - 2r_1\cos\theta_1 z + r_1^2)(z - r_3) = (z^2+0.5111z+0.7363)(z+0.3628)$.
Thus, $2r_1\cos\theta_1 = -0.5111$, $r_1^2 = 0.7363$, $r_3 = 0.3628$, and $\theta_3 = \pi$.
From the above $r_1 = \sqrt{0.7363} = 0.8581$ and $\cos\theta_1 = \dfrac{-0.5111}{2\sqrt{0.7363}} = -0.2978$.

$$\frac{1}{B(z)} = \frac{1}{(z^2+0.5111z+0.7363)(z+0.3628)}$$
$$= \frac{-0.7326 - j\,0.0959}{z+0.25555-j\,0.81914} + \frac{-0.7326 + j0.0959}{z+0.25555+j0.81914} + \frac{1.4652}{z+0.3628}.$$

$\mathbf{P}_1 = \begin{bmatrix}\cos\theta_1 & r_1 & r_1^2\cos\theta_1\end{bmatrix} = \begin{bmatrix}-0.2978 & 0.8581 & -0.2193\end{bmatrix}$,

$\mathbf{Q}_1 = \begin{bmatrix}-\sin\theta_1 & 0 & r_1^2\sin\theta_1\end{bmatrix} = \begin{bmatrix}-0.9546 & 0 & 0.7029\end{bmatrix}$,

$R_1 = 0.40357$, $X_1 = -0.36481$,

$\mathbf{P}_3 = \begin{bmatrix}\cos\theta_3 & r_3 & r_3^2\cos\theta_1\end{bmatrix} = \begin{bmatrix}-1 & 0.3628 & -0.1316\end{bmatrix}$,

$\mathbf{Q}_3 = \begin{bmatrix}-\sin\theta_3 & 0 & r_3^2\sin\theta_3\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\end{bmatrix}$,

$R_3 = 1.4652$, and $X_3 = 0$.

Thus, $\Delta r_1 = (-R_1\mathbf{P}_1 + X_1\mathbf{Q}_1)\cdot\Delta\mathbf{B} = -0.1266\,\Delta b_0 + 0.6286\,\Delta b_1 - 0.2281\Delta b_2$,

$$\Delta\theta_1 = -\frac{1}{r_1}(X_1\mathbf{P}_1 + \mathbf{R}_1\mathbf{Q}_1)\cdot\Delta\mathbf{B} = -0.8483\Delta b_0 + 0.0959\Delta b_1 + 0.5756\,\Delta b_2,$$

$$\Delta r_3 = (-R_3\mathbf{P}_3 + X_3\mathbf{Q}_3)\cdot\Delta\mathbf{B} = -R_3\mathbf{P}_3\cdot\Delta\mathbf{B} = 1.4652\Delta b_0 - 0.5316\Delta b_1 + 0.1929\,\Delta b_2,$$

$$\Delta\theta_3 = -\frac{1}{r_3}(X_3\mathbf{P}_3 + \mathbf{R}_3\mathbf{Q}_3)\cdot\Delta\mathbf{B} = 0.$$

(b) Cascade Form: $B(z) = z^3 + b_2 z^2 + b_1 z + b_0 = (z^2 + c_1 z + c_0)(z + d_0) = B_1(z)B_2(z)$ where

$$B_1(z) = z^2 + c_1 z + c_0 = (z-z_1)(z-z_2) = (z - r_1 e^{j\theta_1})(z - r_1 e^{-j\theta_1}) = z^2 - 2r_1\cos\theta_1 z + r_1^2$$

and $B_2(z) = z + d_0 = z - r_3 e^{j\theta_3}$. Comparing with the denominator of the given transfer function we get $2r_1\cos\theta_1 = -0.5111$, $r_1^2 = 0.7363$, $r_3 = 0.3628$, and $q_3 = \pi$. Hence, $r_1 = 0.8581$ and $\cos\theta_1 = -0.2978$.

Now, $\dfrac{1}{B_1(z)} = \dfrac{-j\,0.6104}{z+0.2556-j0.8191} + \dfrac{j\,0.6104}{z+0.2556+j0.8191}$.

Hence, $R_1 = 0$ and $X_1 = -0.6104$.

$\mathbf{P}_1 = [\cos\theta_1 \quad r_1] = [-0.2978 \quad 0.8581]$, and $\mathbf{Q}_1 = [-\sin\theta_1 \quad 0] = [-0.9546 \quad 0]$. Next, we observe $\dfrac{1}{B_2(z)} = \dfrac{1}{z+0.3628}$. This implies, $R_3 = 1$ and $X_3 = 0$. $\mathbf{P}_3 = \cos\theta_3 = -1$, $\mathbf{Q}_3 = -\sin\theta_3 = 0$. Thus,

$$\Delta r_1 = (-R_1\mathbf{P}_1 + X_1\mathbf{Q}_1)\cdot[\Delta c_0 \quad \Delta c_1]^T = X_1 Q_1 \cdot [\Delta c_0 \quad \Delta c_1]^T = 0.5827\,\Delta c_0,$$

$$\Delta\theta_1 = -\frac{1}{r_1}(X_1\mathbf{P}_1 + \mathbf{R}_1\mathbf{Q}_1)\cdot[\Delta c_0 \quad \Delta c_1]^T = -0.2118\Delta c_0 + 0.6104\Delta c_1$$

$$\Delta r_3 = (-R_3\mathbf{P}_3 + X_3\mathbf{Q}_3)\cdot\Delta d_0 = -R_3\mathbf{P}_3\cdot\Delta d_0 = -\Delta d_0$$

$$\Delta\theta_3 = -\frac{1}{r_3}(X_3\mathbf{P}_3 + \mathbf{R}_3\mathbf{Q}_3)\cdot\Delta d_0 = -\frac{1}{r_3}\mathbf{R}_3\mathbf{Q}_3\cdot\Delta d_0 = 0$$

Example E12.2: Determine the output noise variance due to the propagation of the input quantization noise for the causal IIR digital filter:

$$H(z) = \frac{3(2z+1)(0.5z^2 - 0.3z + 1)}{(3z+1)(4z+1)(z^2 - 0.5z + 0.4)}.$$

Answer: $H_2(z) = \dfrac{3(2z+1)(0.5z^2 - 0.3z+1)}{(3z+1)(4z+1)(z^2 - 0.5z + 0.4)} = \dfrac{-1.7049}{z+\frac{1}{3}} + \dfrac{2.8245}{z+\frac{1}{4}} + \dfrac{-0.86955z + 0.52675}{z^2 - 0.5z + 0.4}$.

Again, from Eq. (12.87) and Table 12.44, we get

$$\sigma_{2,n}^2 = \frac{(1.7049)^2}{1-\left(\frac{1}{3}\right)^2} + \frac{(2.8245)^2}{1-\left(\frac{1}{4}\right)^2} + 2\left(\frac{-1.7049\times 2.8245}{1-\frac{1}{12}}\right)$$
$$+\ 2\times\frac{-1.7049(-0.86955 + 0.52675\times(-1/3))}{1-0.5(-1/3)+0.4(-1/3)^2} \ +\ 2\times\frac{2.8245(-0.86955+0.52675\times(-1/4))}{1-0.5(-1/4)+0.4(-1/4)^2}$$
$$+\ \frac{[(-0.86955)^2 + (0.52675)^2](1-0.4^2) - 2\times(-0.86955)\times 0.52675\times(1-0.4)\times(-0.5)}{(1-0.4^2)^2 + 2(0.4)(-0.5)^2 - (1+0.4^2)\times(-0.5)^2}$$
$$= 3.271 + 8.5097 - 10.5065 + 2.9425 - 4.9183 + 0.9639 = 0.2613.$$

Output of Program 12_4 is 0.26129.

Example E12.3: Realize the causal IIR transfer function

$$H(z) = \frac{(z-2)(z+3)}{(z+0.3)(z-0.4)}$$

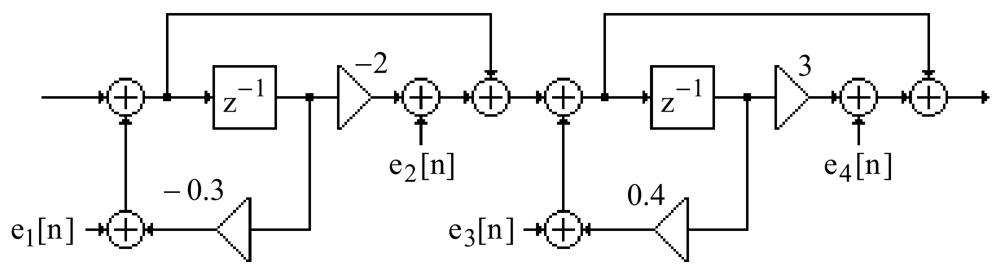
in four different cascade forms with each first-order stage implemented in direct form II.

(a) Show the noise model for each unscaled structure for the computation of the product round-off noise at the output assuming quantization of products before addition assuming fixed-point implementation with either rounding or two's-complement truncation. Compute the normalized output round-off noise variance for each realization. Which cascade realization has the lowest round-off noise?

(b) Repeat part (a) assuming quantization after addition of product.

Answer: (a) Quantization of products before addition.

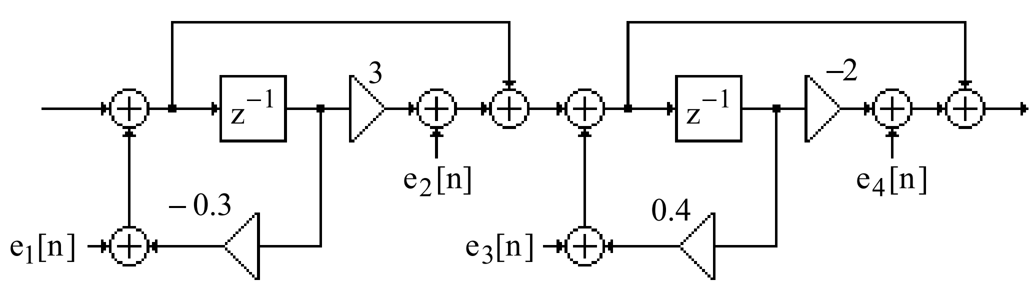
Cascade Structure #1: $H(z) = \left(\frac{1-2z^{-1}}{1+0.3z^{-1}}\right)\left(\frac{1+3z^{-1}}{1-0.4z^{-1}}\right)$. The noise model of this structure is shown below:



The noise transfer function from the noise source $e_1[n]$ to the filter output is $G_1(z) = H(z) = \frac{z^2+z-6}{z^2-0.1z-0.12} = 1 + \frac{-7.7714}{z-0.4} + \frac{8.8714}{z+0.3}$. The corresponding normalized noise variance at the output is $\sigma_1^2 = 1 + \frac{(-7.7714)^2}{1-(0.4)^2} + \frac{(8.8714)^2}{1-(-0.3)^2} + 2\left(\frac{-7.7714\times 8.8714}{1-(0.4)(-0.3)}\right) = 36.271.$

The noise transfer function from the noise sources $e_2[n]$ and $e_3[n]$ to the filter output is $G_2(z) = \frac{z+3}{z-0.4} = 1 + \frac{3.4}{z-0.4}$. The normalized noise variance at the output due to each of these noise sources is $\sigma_2^2 = 1 + \frac{(3.4)^2}{1-(0.4)^2} = 14.762$. The noise transfer function from the noise source $e_4[n]$ to the filter output is $G_3(z) = 1$, The corresponding normalized noise variance at the output is $\sigma_3^2 = 1$. Hence the total noise variance at the output is $\sigma_o^2 = \sigma_1^2 + 2\sigma_2^2 + \sigma_3^2 = 66.795$.

Cascade Structure #2: $H(z) = \left(\frac{1+3z^{-1}}{1+0.3z^{-1}}\right)\left(\frac{1-2z^{-1}}{1-0.4z^{-1}}\right)$. The noise model of this structure is shown below:



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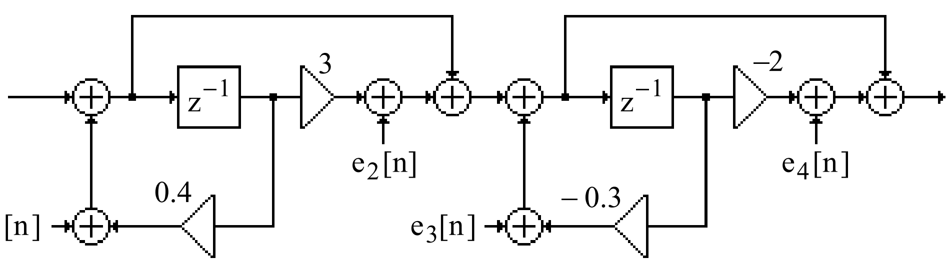
The noise transfer function from the noise source $e_1[n]$ to the filter output is $G_1(z) = H(z) = \dfrac{z^2 + z - 6}{z^2 - 0.1z - 0.12} = 1 + \dfrac{-7.7714}{z - 0.4} + \dfrac{8.8714}{z + 0.3}$. The corresponding normalized noise variance at the output is $\sigma_1^2 = 1 + \dfrac{(-7.7714)^2}{1-(0.4)^2} + \dfrac{(8.8714)^2}{1-(-0.3)^2} + 2\left(\dfrac{-7.7714 \times 8.8714}{1-(0.4)(-0.3)}\right) = 36.271.$

The noise transfer function from the noise sources $e_2[n]$ and $e_3[n]$ to the filter output is $G_2(z) = \dfrac{z-2}{z-0.4} = 1 + \dfrac{-1.6}{z-0.4}$. The normalized noise variance at the output due to each of these noise sources is $\sigma_2^2 = 1 + \dfrac{(-1.6)^2}{1-(0.4)^2} = 4.0476$. The noise transfer function from the noise source $e_4[n]$ to the filter output is $G_3(z) = 1$, The corresponding normalized noise variance at the output is $\sigma_3^2 = 1$. Hence the total noise variance at the output is $\sigma_o^2 = \sigma_1^2 + 2\sigma_2^2 + \sigma_3^2 = 45.366.$

<u>Cascade Structure</u> <u>#3</u>: $H(z) = \left(\dfrac{1+3z^{-1}}{1-0.4z^{-1}}\right)\left(\dfrac{1-2z^{-1}}{1+0.3z^{-1}}\right)$. The noise model of this structure is shown below:



The noise transfer function from the noise source $e_1[n]$ to the filter output is $G_1(z) = H(z) = \dfrac{z^2 + z - 6}{z^2 - 0.1z - 0.12} = 1 + \dfrac{-7.7714}{z - 0.4} + \dfrac{8.8714}{z + 0.3}$. The corresponding normalized noise variance at the output is $\sigma_1^2 = 1 + \dfrac{(-7.7714)^2}{1-(0.4)^2} + \dfrac{(8.8714)^2}{1-(-0.3)^2} + 2\left(\dfrac{-7.7714 \times 8.8714}{1-(0.4)(-0.3)}\right) = 36.271.$

The noise transfer function from the noise sources $e_2[n]$ and $e_3[n]$ to the filter output is $G_2(z) = \dfrac{z-2}{z+0.3} = 1 + \dfrac{-2.3}{z+0.3}$. The normalized noise variance at the output due to each of these noise sources is $\sigma_2^2 = 1 + \dfrac{(-2.3)^2}{1-(-0.3)^2} = 6.8132$. The noise transfer function from the noise source $e_4[n]$ to the filter output is $G_3(z) = 1$, The corresponding normalized noise variance at the output is $\sigma_3^2 = 1$. Hence the total noise variance at the output is $\sigma_o^2 = \sigma_1^2 + 2\sigma_2^2 + \sigma_3^2 = 50.897.$

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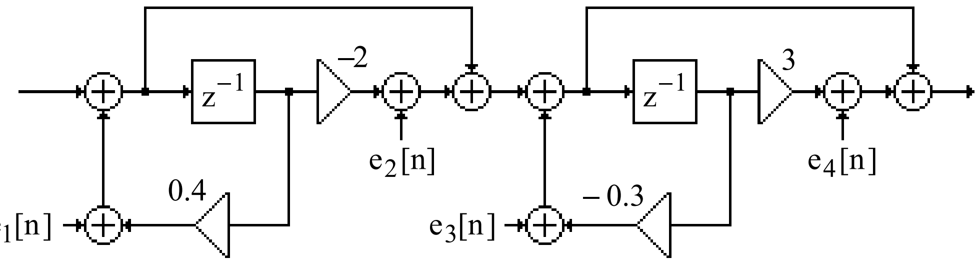
Cascade Structure #4: $H(z) = \left(\dfrac{1-2z^{-1}}{1-0.4z^{-1}}\right)\left(\dfrac{1+3z^{-1}}{1+0.3z^{-1}}\right)$. The noise model of this structure is shown below:



The noise transfer function from the noise source $e_1[n]$ to the filter output is $G_1(z) = H(z) = \dfrac{z^2+z-6}{z^2-0.1z-0.12} = 1 + \dfrac{-7.7714}{z-0.4} + \dfrac{8.8714}{z+0.3}$. The corresponding normalized noise variance at the output is $\sigma_1^2 = 1 + \dfrac{(-7.7714)^2}{1-(0.4)^2} + \dfrac{(8.8714)^2}{1-(-0.3)^2} + 2\left(\dfrac{-7.7714\times 8.8714}{1-(0.4)(-0.3)}\right) = 36.271.$

The noise transfer function from the noise sources $e_2[n]$ and $e_3[n]$ to the filter output is $G_2(z) = \dfrac{z+3}{z+0.3} = 1 + \dfrac{2.7}{z+0.3}$. The normalized noise variance at the output due to each of these noise sources is $\sigma_2^2 = 1 + \dfrac{(2.7)^2}{1-(-0.3)^2} = 9.011$. The noise transfer function from the noise source $e_4[n]$ to the filter output is $G_3(z) = 1$, The corresponding normalized noise variance at the output is $\sigma_3^2 = 1$. Hence the total noise variance at the output is $\sigma_o^2 = \sigma_1^2 + 2\sigma_2^2 + \sigma_3^2 = 55.293.$

Hence, the Cascade Structure #2 has the smallest roundoff-noise variance.

(b) Quantization of products after addition.

From the results of Part (a) we have here the following roundoff noise variances:

Cascade Structure #1: $\sigma_o^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 = 52.033.$

Cascade Structure #2: $\sigma_o^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 = 41.319.$

Cascade Structure #3: $\sigma_o^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 = 44.084.$

Cascade Structure #4: $\sigma_o^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 = 46.282.$

In this case also the Cascade Form 2 has the smallest roundoff-noise variance.

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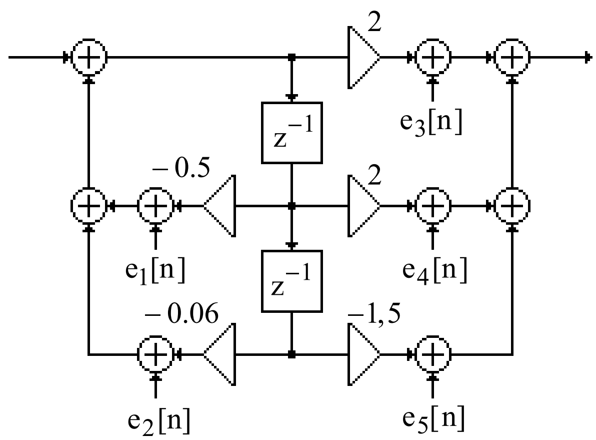
Example E12.4: Realize the causal second-order IIR transfer function

$$H(z) = \frac{2 + 2z^{-1} - 1.5z^{-2}}{1 + 0.5z^{-1} + 0.06z^{-2}}$$

in (1) direct form, (2) cascade form, and (3) parallel form. Each section in the cascade and parallel structures is realized in direct form II. Show the noise model for each unscaled structure for the computation of the product round-off noise at the output assuming quantization of products before addition and assuming fixed-point implementation with either rounding or two's-complement truncation. Compute the product round-off noise variance for each realization. Which realization has the lowest round-off noise?

Answer: (a) The noise model of the Direct Form II realization of H(z) is shown below:



The noise transfer function from the noise sources $e_3[n]$, $e_4[n]$ and $e_5[n]$ to the filter output is $G_2(z) = 1$, and the noise transfer function from the noise sources $e_1[n]$ and $e_2[n]$ to the filter output is $G_1(z) = H(z) = \dfrac{2 + 2z^{-1} - 1.5z^{-2}}{1 + 0.5z^{-1} - 0.06z^{-2}} = 2 + \dfrac{-18.2}{z + 0.2} + \dfrac{19.2}{z + 0.3}$. Using Table 12.4, the normalized noise variance at the output due to each of the noise sources $e_1[n]$ and $e_2[n]$ is then $\sigma_1^2 = 2\left[4 + \dfrac{(-18.2)^2}{1-(-0.2)^2} + \dfrac{(19.2)^2}{1-(-0.3)^2} + 2\left(\dfrac{-18.2 \times 19.2}{1-(-0.2)(-0.3)}\right)\right] = 10.651$, and the normalized noise variance at the output due to each of the noise sources $e_3[n]$, $e_4[n]$ and $e_5[n]$ is $\sigma_2^2 = 1$. Hence the total noise variance at the output is $\sigma_o^2 = 2\sigma_1^2 + 3\sigma_2^2 = 24.302$.

(b) Cascade Form Realization: $H(z) = \dfrac{2(1 - 0.5z^{-1})(1 + 1.5z^{-1})}{(1 + 0.2z^{-1})(1 + 0.3z^{-1})}$. There are more than 2 possible cascade realizations. We consider here only two such structures.

<u>Cascade Form #1</u>: $H(z) = 2\left(\dfrac{1 - 0.5z^{-1}}{1 + 0.2z^{-1}}\right)\left(\dfrac{1 + 1.5z^{-1}}{1 + 0.3z^{-1}}\right)$. The noise model of this realization is shown below:

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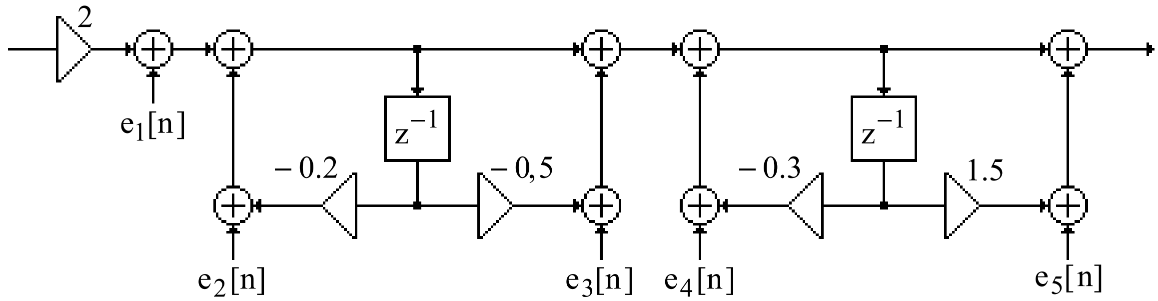
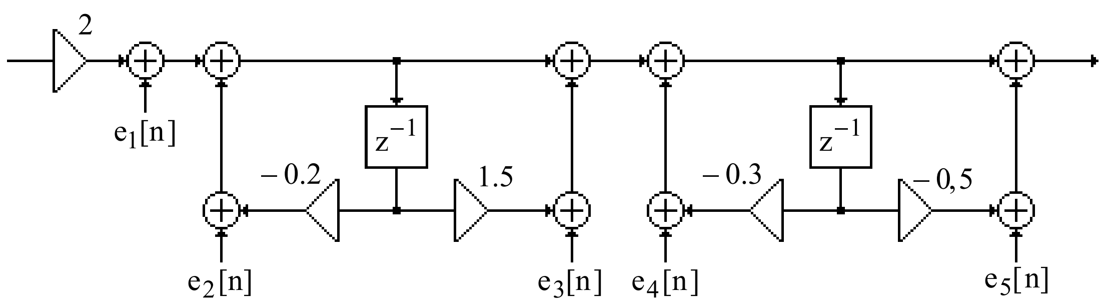


The noise transfer function from the noise sources $e_1[n]$ and $e_2[n]$ to the filter output is $G_1(z) = \dfrac{1+1z^{-1}-0.75z^{-2}}{1+0.5z^{-1}-0.06z^{-2}} = 1 + \dfrac{-9.1}{z+0.2} + \dfrac{9.6}{z+0.3}$. Its contribution to the output noise variance is $\sigma_1^2 = 1 + \dfrac{(-9.1)(-9.1)}{1-(-0.2)(-0.2)} + \dfrac{(9.6)(9.6)}{1-(-0.3)(-0.3)} + 2\left[\dfrac{(-9.1)(9.6)}{1-(-0.2)(-0.3)}\right] = 2.6628$.

The noise transfer function from the noise sources $e_3[n]$ and $e_4[n]$ to the filter output is $G_2(z) = \dfrac{1+1.5z^{-1}}{1+0.3z^{-1}} = 1 + \dfrac{1.2}{z+0.3}$. Its contribution to the output noise variance is $\sigma_2^2 = 1 + \dfrac{(1.2)^2}{1-(-0.3)^2} = 2.5824$. Finally, the noise transfer function from the noise source $e_5[n]$ to the filter output is $G_3(z) = 1$. Its contribution to the output noise variance is $\sigma_3^2 = 1$. Hence the total noise variance at the output is
$\sigma_o^2 = 2\sigma_1^2 + 2\sigma_2^2 + \sigma_3^2 = 2(2.6628) + 2(2.5824) + 1 = 11.49.$

Cascade Form #2: $H(z) = 2\left(\dfrac{1+1.5z^{-1}}{1+0.2z^{-1}}\right)\left(\dfrac{1-0.5z^{-1}}{1+0.3z^{-1}}\right)$. The noise model of this realization is shown below:



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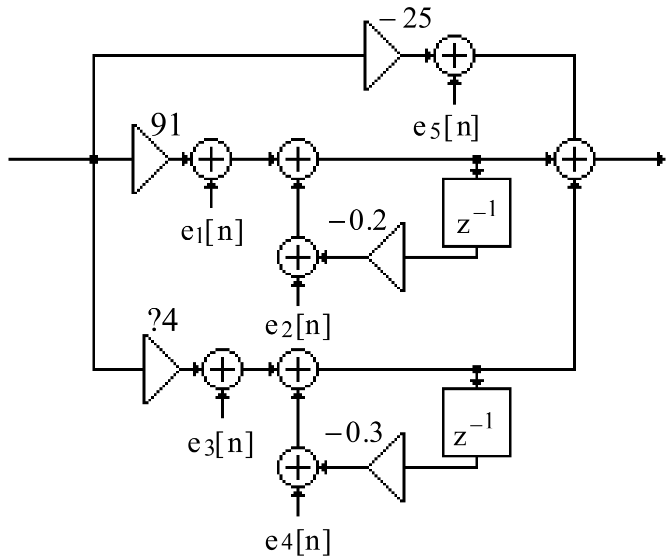
The noise transfer function from the noise sources $e_1[n]$ and $e_2[n]$ to the filter output is $G_1(z) = \dfrac{1 + 1z^{-1} - 0.75z^{-2}}{1 + 0.5z^{-1} - 0.06z^{-2}} = 1 + \dfrac{-9.1}{z + 0.2} + \dfrac{9.6}{z + 0.3}$. Its contribution to the output noise variance is $\sigma_1^2 = 1 + \dfrac{(-9.1)(-9.1)}{1 - (-0.2)(-0.2)} + \dfrac{(9.6)(9.6)}{1 - (-0.3)(-0.3)} + 2\left[\dfrac{(-9.1)(9.6)}{1 - (-0.2)(-0.3)}\right] = 2.6628$.

The noise transfer function from the noise sources $e_3[n]$ and $e_4[n]$ to the filter output is $G_2(z) = \dfrac{1 - 0.5z^{-1}}{1 + 0.3z^{-1}} = 1 + \dfrac{-0.8}{z + 0.3}$. Its contribution to the output noise variance is $\sigma_2^2 = 1 + \dfrac{(-0.8)^2}{1 - (-0.3)^2} = 1.7033$, Finally, the noise transfer function from the noise source $e_5[n]$ to the filter output is $G_3(z) = 1$. Its contribution to the output noise variance is $\sigma_3^2 = 1$. Hence the total noise variance at the output is $\sigma_o^2 = 2\sigma_1^2 + 2\sigma_2^2 + \sigma_3^2 = 2(2.6628) + 2(1.7033) + 1 = 9.7322$.

(c) Parallel Form I Realization: $H(z) = -25 + \dfrac{91}{1 + 0.2z^{-1}} + \dfrac{-64}{1 + 0.3z^{-1}}$. The noise model of this realization is shown below:



The noise transfer function from the noise sources $e_1[n]$ and $e_2[n]$ to the filter output is $G_1(z) = \dfrac{1}{1 + 0.2z^{-1}} = 1 + \dfrac{-0.2}{z + 0.2}$. Its contribution to the output noise variance is

$$\sigma_1^2 = 1 + \frac{(-0.2)^2}{1 - (-0.2)^2} = 1.0417.$$

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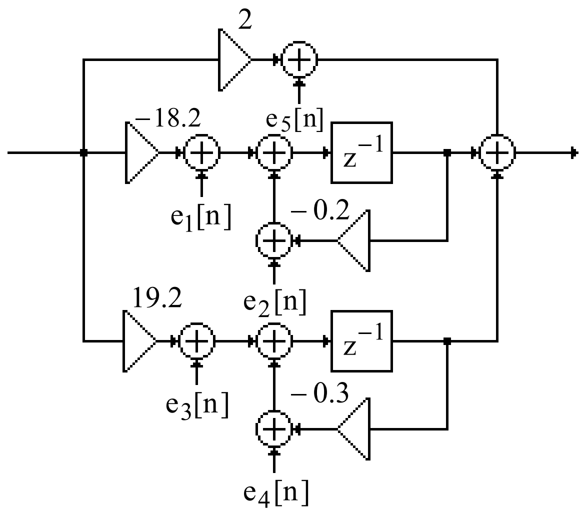
The noise transfer function from the noise sources $e_3[n]$ and $e_4[n]$ to the filter output is $G_2(z) = \dfrac{1}{1+0.3z^{-1}} = 1 + \dfrac{-0.3}{z+0.3}$. Its contribution to the output noise variance is $\sigma_2^2 = 1 + \dfrac{(-0.3)^2}{1-(-0.3)^2} = 1.0989.$

Finally, the noise transfer function from he noise sources $e_5[n]$ to the filter output is $G_3(z) = 1$. Its contribution to the output noise variance is $\sigma_3^2 = 1$. Hence the total noise variance at the output is $\sigma_o^2 = 2\sigma_1^2 + 2\,\sigma_2^2 + \sigma_3^2 = 5.2812.$

Parallel Form II Realization: $H(z) = 2 + \dfrac{-18.2}{z+0.2} + \dfrac{19.2}{z+0.3}$. The noise model of this realization is shown below:



The noise transfer function from the noise sources $e_1[n]$ and $e_2[n]$ to the filter output is $G_1(z) = \dfrac{z^{-1}}{1+0.2z^{-1}} = \dfrac{1}{z+0.2}$. Its contribution to the output noise variance is $\sigma_1^2 = \dfrac{1}{1-(-0.2)^2} = 1.0417.$

The noise transfer function from the noise sources $e_3[n]$ and $e_4[n]$ to the filter output is $G_2(z) = \dfrac{z^{-1}}{1+0.3z^{-1}} = \dfrac{1}{z+0.3}$. Its contribution to the output noise variance is $\sigma_2^2 = \dfrac{1}{1-(-0.3)^2} = 1.0989.$

Finally, the noise transfer function from the noise sources $e_5[n]$ and $e_5[n]$ to the filter output is $G_3(z) = 1$. Its contribution to the output noise variance is $\sigma_3^2 = 1$. Hence the total noise variance at the output is $\sigma_o^2 = 2\sigma_1^2 + 2\,\sigma_2^2 + \sigma_3^2 = 5.2812.$

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As a result, both Parallel Form structures have the smallest product roundoff noise variance.

Example E12.5: One possible two-multiplier realization of a second-order Type 2 allpass transfer function

$$A_2(z) = \frac{d_1 d_2 + d_1 z^{-1} + z^{-2}}{1 + d_1 z^{-1} + d_1 d_2 z^{-2}}$$

is shown in Figure E12.1. Derive the expression for the normalized steady-state output noise variance due to product round-off assuming fixed-point implementation with either rounding or two's-complement truncation.

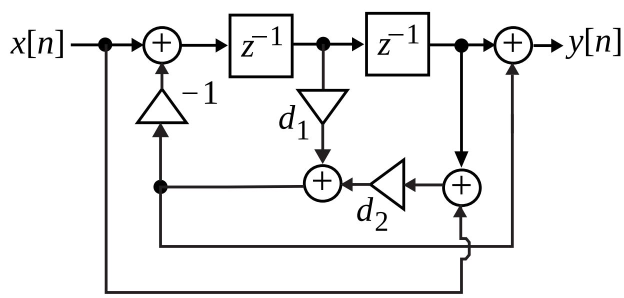
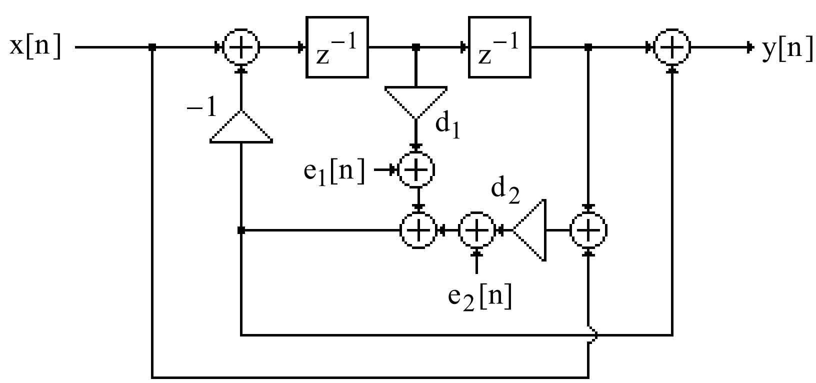


Figure E12.1

Answer: The noise model of Figure E12.1 is shown below:



The noise transfer function from the noise sources $e_1[n]$ and $e_2[n]$ to the output is given by

$$G_1(z) = \frac{1 - z^{-2}}{1 + d_1 z^{-1} + d_2 z^{-2}} = \frac{z^2 - 1}{z^2 + d_1 z + d_2} = 1 - \frac{d_1 z + (1 + d_2)}{z^2 + d_1 z + d_2}$$

Using Table 12.4 we arrive at the total normalized output noise power as

$$\sigma_n^2 = 2\left[1 + \frac{\left(d_1^2 + (1+d_2)^2\right)(1 - d_2^2) - 2\left(d_1(1+d_2) - d_1(1+d_2)d_2\right)d_1}{(1-d_2^2)^2 + 2 d_2 d_1^2 - (1 + d_2^2)\, d_1^2}\right]$$

$$= 2\left[1 + \frac{(1-d_2^2)\left((1+d_2)^2 - d_1^2\right)}{(1-d_2)^2\left((1+d_2)^2 - d_1^2\right)}\right] = \frac{4}{1-d_2}.$$

Example E12.6: Scale the first-order digital filter structure of Figure E12.2 using the L_2-norm scaling rule.

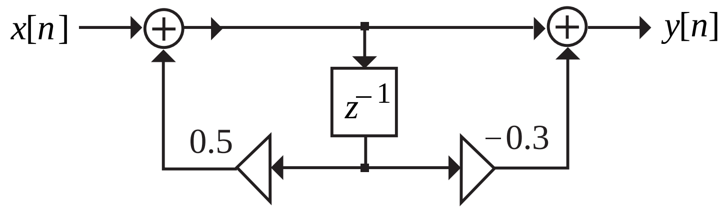
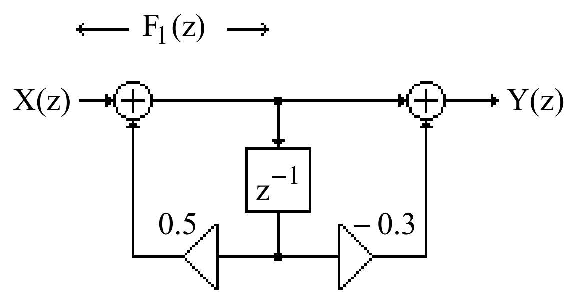


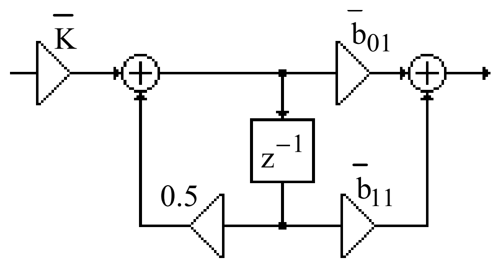
Figure E12.2

Answer: The unscaled structure is shown below.



Now, $F_1(z) = \dfrac{1}{1-0.5z^{-1}} = 1 + \dfrac{0.5}{z-0.5}$. Using Table 12.4 we obtain $\|F_1\|_2^2 = 1 + \dfrac{(0.5)^2}{1-(0.5)^2} = 1.3333$. Next, $H(z) = \dfrac{1-0.3z^{-1}}{1-0.5z^{-1}} = \dfrac{z-0.3}{z-0.5} = 1 + \dfrac{0.2}{z-0.5}$. Using Table 12.4 we obtain $\|H\|_2^2 = 1 + \dfrac{(0.2)^2}{1-(0.5)^2} = 1.0533$. From Eq. (12.128a), $\|F_1\|_2 = \alpha_1 = 1.1547$, and from Eq. (12.128b), $\|H\|_2 = \alpha_2 = \sqrt{1.0533} = 1.0263$.

The scaled structure is shown below, where $\bar{b}_{01} = \beta_1$ and $\bar{b}_{11} = -0.3\beta_1$. From Eqs. (12.129) and (12.132a), $\bar{K} = \beta_0 K = \beta_0 = \dfrac{1}{\alpha_1} = 0.86603$, and from Eq. (12.132b), $\beta_1 = \dfrac{\alpha_1}{\alpha_2} = \dfrac{1.1547}{1.0263} = 1.1251$. Therefore, $\bar{b}_{01} = \beta_1 = 1.1251$, and $\bar{b}_{11} = -0.3\beta_1 = -0.3375$.



Example E12.7: Scale the first-order digital filter structure of Figure E12.3 using the L_2-norm scaling rule.

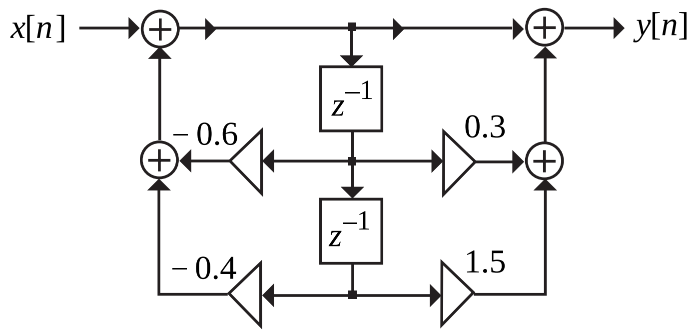
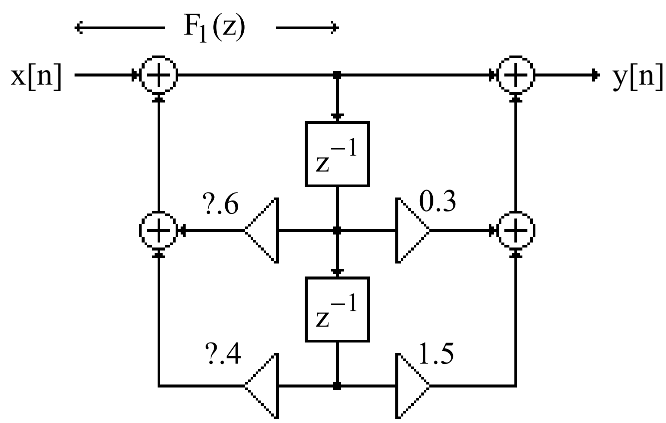


Figure E12.3

Answer: The unscaled structure is shown below.



Here, $F_1(z) = \dfrac{1}{1 + 0.6z^{-1} + 0.4z^{-2}} = \dfrac{z^2}{z^2 + 0.6\,z + 0.4} = 1 - \dfrac{0.6z + 0.4}{z^2 + 0.6z + 0.4}$. Using Table 12.4, we get $\|F_1\|_2^2 = 1 + \dfrac{[0.6^2 + 0.4^2][1 - 0.4^2] - 2 \times (1 - 0.4) \times 0.6 \times 0.4 \times 0.8}{[1 - 0.4^2]^2 + 2 \times 0.4 \times (0.6)^2 - [1 + (0.4)^2] \times (0.6)^2} = 1.4583$. Next, we note

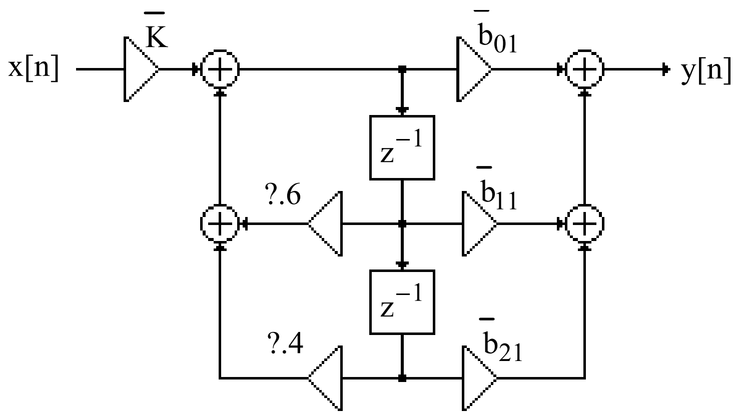
$$H(z) = \frac{1 + 0.3z^{-1} + 1.5z^{-2}}{1 + 0.6z^{-1} + 0.4z^{-2}} = \frac{z^2 + 0.3z + 1.5}{z^2 + 0.6z + 0.4} = 1 + \frac{-0.3z - 1.1}{z^2 + 0.6z + 0.4}.$$

Using Table 12.4 we then get

$$\|H\|_2^2 = 1 + \frac{[0.3^2 + 1.1^2][1 - 0.4^2] - 2 \times 0.6 \times 0.3 \times (-1.1) \times (1 - 0.4)}{[1 - 0.4^2]^2 + 2 \times 0.4 \times (0.6)^2 - [1 + 0.4^2] \times (0.6)^2} = 3.3083.$$

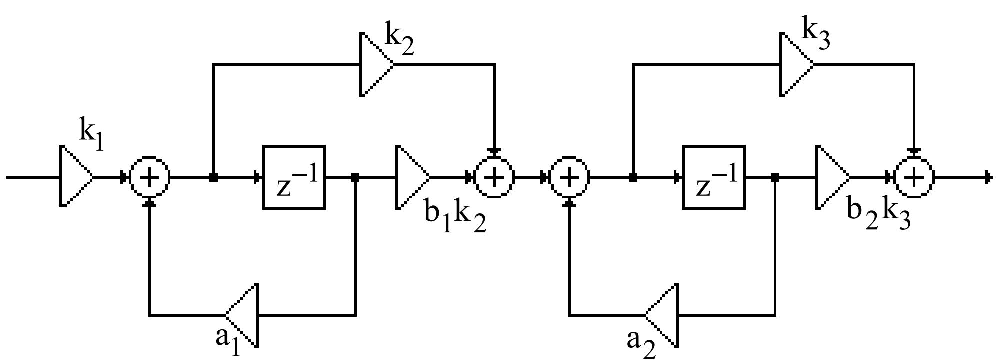
From Eq. (12.128a), $\|F_1\|_2 = \alpha_1 = \sqrt{1.4583}$, and $\|H\|_2 = \alpha_2 = \sqrt{3.3083}$. Hence, from Eq. (12.132a), $\beta_0 = \frac{1}{\alpha_1} = \frac{1}{\sqrt{1.4583}} = 0.8281$, and from Eq. (12.132b), $\beta_1 = \frac{\alpha_1}{\alpha_2} = \frac{\sqrt{1.4583}}{\sqrt{3.3083}} = 0.6639.$

The scaled structure is as shown below, where $\overline{K} = \beta_0 K = \beta_0 = 0.8281$, $\overline{b}_{01} = (1)(\beta_1) = 0.6639$, $\overline{b}_{11} = (0.3)(\beta_1) = 0.1992$, and $\overline{b}_{21} = (1.5)(\beta_1) = 0.9959$.



Example E12.8: Scale the structures realized in Example E12.3 using the L_2-norm scaling rule and then compute the output noise variances due to product round-off assuming quantization of products before addition. What would be the output noise variances if quantization is carried out after addition?

Answer: (a) The scaled structure is shown below. The value of the scaling constants is found below.



Additional Examples of Chapter 12:
Analysis of Finite Wordlength Effects

Cascade Structure #1: $H(z)=\left(\frac{1-2z^{-1}}{1+0.3z^{-1}}\right)\left(\frac{1+3z^{-1}}{1-0.4z^{-1}}\right)$. Here, $a_1=-0.3$, $b_1=-2$, $a_2=0.4$, and $b_2=3$.

$F_1(z)=\frac{z^{-1}}{1+0.3z^{-1}}=\frac{1}{z+0.3}$. Thus, using Program 12_4 we get $\|F_1\|_2^2=1.9089$. Hence, $\gamma_1=\|F_1\|_2=\sqrt{1.9089}=1.0483$. $F_2(z)=\frac{1-2z^{-1}}{1+0.3z^{-1}}\cdot\frac{z^{-1}}{1-0.4z^{-1}}=\frac{z-2}{z^2-0.1z-0.12}$. Using Program 12_4 we get $\|F_2\|_2^2=4.6722$. Hence, $\gamma_2=\|F_2\|_2=\sqrt{4.6722}=2.1615$. Next, $H(z)=\frac{z^2+z-6}{z^2-0.1z-0.12}$. Using Program 12_4 we get $\|H\|_2^2=36.271$. Hence, $\gamma_0=\|H\|_2=\sqrt{36.271}=6.0226$.

The scaling multipliers are therefore given by $k_1=\frac{1}{\gamma_1}=0.95393$, $k_2=\frac{\gamma_1}{\gamma_2}=0.48499$, $k_3=\frac{\gamma_2}{\gamma_0}=0.3589$. $b_1k_2=-0.96998$, and $b_2k_3=1.0767$.

The noise at the output due to the scaling constant k_1 and multiplier a_1 have a variance $\sigma_1^2=\gamma_1^2=1.9089$.

Noise at the output due to a_2, k_2 and b_1k_2 have variance σ_2^2 which is calculated below. The noise transfer function for these noise sources is

$G_2(z)=\frac{0.3589+1.0767z^{-1}}{1-0.4z^{-1}}=\frac{0.3589z+1.0767}{z-0.4}$. Using Program 12_4 we get $\sigma_2^2=1.1654$..

Hence the total noise power (variance) at the output = $2\times 1.9089+3\times 1.1654+2=9.314$.

In case quantization is carried out after addition, then the total noise power at the output = 1.9089 + 1.1654 + 1 = 4.0743.

Cascade Structure #2: $H(z)=\left(\frac{1+3z^{-1}}{1+0.3z^{-1}}\right)\left(\frac{1-2z^{-1}}{1-0.4z^{-1}}\right)$. Here $a_1=-0.3$, $b_1=3$, $a_2=0.4$, and $b_2=-2$.

$F_1(z)=\frac{z^{-1}}{1+0.3z^{-1}}=\frac{1}{z+0.3}$. Thus, using Program 12_4 we get $\|F_1\|_2^2=1.9089$. Hence, $\gamma_1=\|F_1\|_2=\sqrt{1.9089}=1.0483$. $F_2(z)=\frac{1+3z^{-1}}{1+0.3z^{-1}}\cdot\frac{z^{-1}}{1-0.4z^{-1}}=\frac{z+3}{z^2-0.1z-0.12}$. Using Program 12_4 we get $\|F_2\|_2^2=10.98$. Hence, $\gamma_2=\|F_2\|_2=\sqrt{10.98}=3.3136$. Next, $H(z)=\frac{z^2+z-6}{z^2-0.1z-0.12}$. Using Program 12_4 we get $\|H\|_2^2=36.271$. Hence, $\gamma_0=\|H\|_2=\sqrt{36.271}=6.0226$.

The scaling multipliers are therefore given by $k_1 = \dfrac{1}{\gamma_1} = 0.95393$, $k_2 = \dfrac{\gamma_1}{\gamma_2} = 0.31636$, $k_3 = \dfrac{\gamma_2}{\gamma_0} = 0.55019$. $b_1 k_2 = 0.94908$, and $b_2 k_3 = -1.1004$.

The noise at the output due to the scaling constant k_1 and multiplier a_1 have a variance $\sigma_1^2 = \gamma_1^2 = 1.9089$.

Noise at the output due to a_2, k_2 and $b_1 k_2$ have variance σ_2^2 which is calculated below. The noise transfer function for these noise sources is

$G_2(z) = \dfrac{0.55019 - 1.1004\, z^{-1}}{1 - 0.4\, z^{-1}} = \dfrac{0.55019 z - 1.1004}{z - 0.4}$. Using Program 12_4 we get $\sigma_2^2 = 1.2253$.

Hence the total noise power (variance) at the output = 2× 1.9089 + 3× 1.2253 + 2 = 9.4937.

In case quantization is carried out after addition, then the total noise power at the output = 1.9089 + 1.2253 + 1 = 4.1342.

<u>Cascade Structure #3</u>: $H(z) = \left(\dfrac{1 + 3z^{-1}}{1 - 0.4z^{-1}}\right)\left(\dfrac{1 - 2\, z^{-1}}{1 + 0.3\, z^{-1}}\right)$. Here $a_1 = 0.4$, $b_1 = 3$, $a_2 = -0.3$, and $b_2 = -2$.

$F_1(z) = \dfrac{z^{-1}}{1 - 0.4 z^{-1}} = \dfrac{1}{z - 0.4}$. Thus, using Program 9_4 we get $\|F_1\|_2^2 = 1.1905$. Hence,

$\gamma_1 = \|F_1\|_2 = \sqrt{1.1905} = 1.0911$. $F_2(z) = \dfrac{1 + 3z^{-1}}{1 - 0.4 z^{-1}} \cdot \dfrac{z^{-1}}{1 + 0.3 z^{-1}} = \dfrac{z + 3}{z^2 - 0.1z - 0.12}$. Using Program 9_4 we get $\|F_2\|_2^2 = 10.98$. Hence, $\gamma_2 = \|F_2\|_2 = \sqrt{10.98} = 3.3136$. Next,

$H(z) = \dfrac{z^2 + z - 6}{z^2 - 0.1z - 0.12}$. Using Program 12_4 we get $\|H\|_2^2 = 36.271$. Hence,

$\gamma_0 = \|H\|_2 = \sqrt{36.271} = 6.0226$.

The scaling multipliers are therefore given by

$k_1 = \dfrac{1}{\gamma_1} = 0.91652$, $k_2 = \dfrac{\gamma_1}{\gamma_2} = 0.32928$, $k_3 = \dfrac{\gamma_2}{\gamma_0} = 0.55019$. $b_1 k_2 = 0.98784$, and $b_2 k_3 = -1.1004$.

The noise at the output due to the scaling constant k_1 and multiplier a_1 have a variance $\sigma_1^2 = \gamma_1^2 = 1.1905$.

Noise at the output due to a_2, k_2 and $b_1 k_2$ have variance σ_2^2 which is calculated below. The noise transfer function for these noise sources is

$G_2(z) = \dfrac{0.55019 - 1.1004 z^{-1}}{1 + 0.3 z^{-1}} = \dfrac{0.55019\, z - 1.1004}{z + 0.3}$. Using Program 12_4 we get $\sigma_2^2 = 2.0625$.

Hence the total noise power (variance) at the output = 2× 1.1905 + 3× 2.0625 + 2 = 10.568.

Additional Examples of Chapter 12: Analysis of Finite Wordlength Effects

In case quantization is carried out after addition, then the total noise power at the output = 1.1905 + 2.0625 + 1 = 4.253.

Cascade Structure #4: $H(z)=\left(\frac{1-2z^{-1}}{1-0.4z^{-1}}\right)\left(\frac{1+3z^{-1}}{1+0.3z^{-1}}\right)$. Here $a_1=0.4$, $b_1=-2$, $a_2=-0.3$, and $b_2=3$.

$F_1(z)=\frac{z^{-1}}{1-0.4z^{-1}}=\frac{1}{z-0.4}$. Thus, using Program 12_4 we get $\|F_1\|_2^2=1.1905$. Hence, $\gamma_1=\|F_1\|_2=\sqrt{1.1905}=1.0911$. $F_2(z)=\frac{1-2z^{-1}}{1-0.4z^{-1}}\cdot\frac{z^{-1}}{1+0.3z^{-1}}=\frac{z-2}{z^2-0.1z-0.12}$. Using Program 12_4 we get $\|F_2\|_2^2=4.6722$. Hence, $\gamma_2=\|F_2\|_2=\sqrt{4.6722}=2.1615$. Next, $H(z)=\frac{z^2+z-6}{z^2-0.1z-0.12}$. Using Program 12_4 we get $\|H\|_2^2=36.271$. Hence, $\gamma_0=\|H\|_2=\sqrt{36.271}=6.0226$.

The scaling multipliers are therefore given by $k_1=\frac{1}{\gamma_1}=0.91651$, $k_2=\frac{\gamma_1}{\gamma_2}=0.50479$, $k_3=\frac{\gamma_2}{\gamma_0}=0.3589$. $b_1k_2=-1.0096$, and $b_2k_3=1.0767$. The noise at the output due to the scaling constant k_1 and multiplier a_1 have a variance $\sigma_1^2=\gamma_1^2=1.1905$.

The noise at the output due to the scaling constant $b_2k_3=1.0767$. and multiplier a_1 have a variance $\sigma_1^2=\gamma_1^2=1.1905$.

Noise at the output due to a_2, k_2 and b_1k_2 have variance σ_2^2 which is calculated below. The noise transfer function for these noise sources is $G_2(z)=\frac{0.3589+1.0767z^{-1}}{1-0.4z^{-1}}=\frac{0.3589z+1.0767}{z-0.4}$. Using Program 12_4 we get $\sigma_2^2=1.9015$. Hence the total noise power (variance) at the output = $2\times$ 1.1905 + $3\times$ 1,9015 + 2 = 10.085.

In case quantization is carried out after addition, then the total noise power at the output = 1.1905 + 1.9105 + 1 = 4.092.

Example E12.9: (a) What is the optimum pole-zero pairing and ordering of the transfer function

$$H(z)=\frac{(z^2+0.8z+0.2)(z^2+0.2z+0.9)(z^2+0.3z+0.5)}{(z^2+0.1z+0.8)(z^2+0.2z+0.4)(z^2+0.6z+0.3)}$$

for obtaining the smallest peak output noise due to product round-off under an L_2-scaling rule?

(b) Repeat part (a) if the objective is to minimize the output noise power due to product round-off under an L_∞-scaling rule.

Answer: First we pair the poles closest to the unit circle with their nearest zeros resulting in the second-order section $H_a(z) = \dfrac{z^2 + 0.2z + 0.9}{z^2 + 0.1z + 0.8}$. Next, the poles that are closest to the poles of $H_a(z)$ are matched with their nearest zeros resulting in the second-order section $H_b(z) = \dfrac{z^2 + 0.3z + 0.5}{z^2 + 0.2z + 0.4}$. Finally, the remaining poles and zeros are matched yielding the second-order section $H_c(z) = \dfrac{z^2 + 0.8z + 0.2}{z^2 + 0.6z + 0.3}$.

For ordering the sections to yield the smallest peak output noise due to product round-off under an L_2-scaling rule, the sections should be placed from most peaked to least peaked as shown below.

$$\longrightarrow \boxed{H_a(z)} \longrightarrow \boxed{H_b(z)} \longrightarrow \boxed{H_c(z)} \longrightarrow$$

For ordering the sections to yield the smallest peak output noise power due to product round-off under an L_∞-scaling rule, the sections should be placed from least peaked to most peaked as shown below.

$$\longrightarrow \boxed{H_c(z)} \longrightarrow \boxed{H_b(z)} \longrightarrow \boxed{H_a(z)} \longrightarrow$$
