Additional Examples of Chapter 7:
LTI Discrete-Time Systems in the Transform Domain

Example E7.1: Consider the following causal IIR transfer function:

$$H(z) = \frac{3z^3 + 2z^2 + 5}{(0.5z+1)(z^2 + z + 0.6)}.$$

Is H(z) a stable transfer function? If it is not stable, find a stable transfer function G(z) such that $|G(e^{j\omega})| = |H(e^{j\omega})|$. Is there any other transfer function having the same magnitude response as that of H(z)?

Answer: $D(z) = (0.5z+1)(z^2 + z + 0.6) = 0.5(z+2)(z + 0.5 - j0.5916)(z + 0.5 + j0.5916)$. Since one of the roots of D(z) is outside the unit circle at z = – 2, H(z) is unstable. To arrive at a stable, transfer function G(z) such that $|G(e^{j\omega})| = |H(e^{j\omega})|$, we multiply H(z) with an allpass function $A(z) = \dfrac{0.5z+1}{z+0.5}$. Hence

$$G(z) = H(z)A(z) = \left(\frac{3z^3 + 2z^2 + 5}{(0.5z+1)(z^2 + z + 0.6)}\right)\left(\frac{0.5z+1}{z+0.5}\right) = \frac{3z^3 + 2z^2 + 5}{(z+0.5)(z^2 + z + 0.6)}.$$

Now, H(z) multiplied with any allpass transfer function will have the same magnitude as $|H(e^{j\omega})|$. Hence, there are infinite number of such transfer functions.

Example E7.2: Determine the location of the notch frequency of the FIR notch filter given by $H(z) = 1 + \sqrt{2}z^{-1} + z^{-2}$.

Answer: The transfer function of the simplest notch filter is given by

$$G(z) = (1 - e^{j\omega_o}z^{-1})(1 - e^{-j\omega_o}z^{-1}) = 1 - 2\cos\omega_o z^{-1} + z^{-2}.$$

In the steady-state, the output for an input $x[n] = \cos\omega_o n$ is given by (see Eq. (3.102))

$$y[n] = |G(e^{j\omega_o})|\cos(\omega_o n + \theta(\omega_o)),$$

Comparing $H(z) = 1 + \sqrt{2}z^{-1} + z^{-2}$. with G(z) as given above we conclude $\cos\omega_o = -1/\sqrt{2}$, i.e., $\omega_o = \cos^{-1}(-1/\sqrt{2})$. Here now $H(e^{j\omega}) = 1 + \sqrt{2}\,e^{-j\omega} + e^{-j2\omega} = \left(2\cos\omega_o + \sqrt{2}\right)e^{-j\omega} = 0$. Hence y[n] = 0.

Example E7.3: A length-10 Type 2 real-coefficient FIR filter has the following zeros: $\lambda_1 = 3$, $\lambda_2 = j0.8$, and $\lambda_3 = j$. (a) Determine the locations of the remaining zeros. (b) What is the transfer function H(z) of the filter?

Answer: The remaining zeros are at $\lambda_4 = 1/\lambda_1 = 1/3$, $\lambda_5 = \lambda_2{}^* = -j0.8$, $\lambda_6 = \lambda_3{}^* = -j$, $\lambda_7 = 1/\lambda_2 = -j0.125$, $\lambda_8 = \lambda_7{}^* = j0.125$, and $\lambda_9 = -1$. Hence, $H(z) = \prod_{i=1}^{9}(1 - \lambda_i z^{-i})$

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$= 1 - 2.3333z^{-1} + 0.8692z^{-2} - 6.4725z^{-3} - 4.27z^{-4} - 4.27z^{-5} - 6.4725z^{-6}$
$+0.8692z^{-7} - 2.3333z^{-8} + z^{-8}$.

Example E7.4: The first 4 impulse response samples of a causal linear-phase FIR transfer function H(z) are given by h[0] = 2, h[1] = –5, h[2] = 0, and h[3] = 3. Determine the remaining impulse response samples of H(z) of lowest order for each type of linear-phase filter.

Answer: (a) Type 1: $\{h[n]\} = \{2,\ -5,\ 0,\ 3,\ 0,\ -5,\ 2\}$,
(b) Type 2: $\{h[n]\} = \{2,\ -5,\ 0,\ 3,\ 3,\ 0,\ -5,\ 2\}$,
(c) Type 3: $\{h[n]\} = \{2,\ -5,\ 0,\ 3,\ 0,\ -3,\ 0,\ 5,\ -2\}$,
(d) Type 4: $\{h[n]\} = \{2,\ -5,\ 0,\ 3,\ -3,\ 0,\ 5,\ -2\}$.

Example E7.5: Design a first-order highpass filter with a normalized 3-dB cutoff frequency at 0.25 radian/sample.

Answer: From Eq. (7.73b) we get $\alpha = \dfrac{1 - \sin(0.25)}{\cos(0.25)} = 0.7767$. Substituting this value of α in Eq. (7.74) we arrive at $H_{HP}(z) = \dfrac{0.8884(1 - z^{-1})}{1 - 0.7767z^{-1}}$.

Example E7.6: Design a second-order bandpass filter with a center frequency at $\omega_o = 0.6\pi$ and a 3-dB bandwidth of $B_w = 0.15\pi$.

Answer: Using Eq. (7.76) we get $\beta = \cos(0.6\pi) = -0.3090$. Next from Eq. (7.78) we get $\dfrac{2\alpha}{1+\alpha^2} = \cos(0.15\pi) = 0.8910$ or, equivalently $\alpha^2 - 2.2447\alpha + 1 = 0$. Solution of this quadratic equation yields $\alpha = 1.6319$ and $\alpha = 0.6128$. Substituting $\alpha = 1.6319$ and $\beta = -0.3090$ in Eq. (7.77) we arrive at the denominator polynomial of the transfer function $H_{BP}(z)$ as
$D(z) = (1 + 0.8133z^{-1} + 1.6319z^{-2}) = \left(1 + (0.4066 + j1.2110)z^{-1}\right)\left(1 + (0.4066 - j1.2110)z^{-1}\right)$
which has roots outside the unit circle making $H_{BP}(z)$ an unstable transfer function.

Next, substituting $\alpha = 0.6128$ and $\beta = -0.3090$ in Eq. (7.77) we arrive at the denominator polynomial of the transfer function $H_{BP}(z)$ as
$D(z) = (1 + 0.4984z^{-1} + 0.6128z^{-2}) = \left(1 + (0.1181 + j0.7040)z^{-1}\right)\left(1 + (0.1181 - j0.7040)z^{-1}\right)$
which has roots inside the unit circle making $H_{BP}(z)$ a stable transfer function. The desired transfer function is therefore $H_{BP}(z) = \dfrac{0.1936(1 - z^{-2})}{1 + 0.4984z^{-1} + 6128z^{-2}}$.

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Example E7.7: Design a second-order bandstop filter with a notch frequency at $\omega_o = 0.4\pi$ and a 3-dB notch bandwidth of $B_w = 0.15\pi$.

Answer: Using Eq. (7.76) we get $\beta = \cos(0.6\pi) = 0.3090$. Next from Eq. (7.77) we get $\frac{2\alpha}{1+\alpha^2} = \cos(0.15\pi) = 0.8910$ or, equivalently $\alpha^2 - 2.2447\alpha + 1 = 0$. Solution of this quadratic equation yields $\alpha = 1.6319$ and $\alpha = 0.6128$. Substituting $\alpha = 1.6319$ and $\beta = 0.3090$ in Eq. (7.80) we arrive at the denominator polynomial of the transfer function $H_{BS}(z)$ as

$$D(z) = (1 + 0.8133z^{-1} + 1.6319z^{-2}) = \left(1 + (0.4066 + j1.2110)z^{-1}\right)\left(1 + (0.4066 - j1.2110)z^{-1}\right)$$

which has roots outside the unit circle making $H_{BS}(z)$ an unstable transfer function.

Next, substituting $\alpha = 0.6128$ and $\beta = 0.3090$ in Eq. (7.80) we arrive at the denominator polynomial of the transfer function $H_{BS}(z)$ as

$$D(z) = (1 - 0.4984z^{-1} + 0.6128z^{-2}) = \left(1 + (0.2492 + j0.7421)z^{-1}\right)\left(1 + (0.2492 - j0.7421)z^{-1}\right)$$

which has roots inside the unit circle making $H_{BS}(z)$ a stable transfer function. The desired transfer function is therefore $H_{BS}(z) = \dfrac{0.8064(1 - 0.4984z^{-1} + z^{-2})}{1 - 0.4984\,z^{-1} + 6128\,z^{-2}}$.

Example E7.8: Show that

$$H(z) = \frac{0.1 + 0.5z^{-1} + 0.45z^{-2} + 0.45z^{-3} + 0.5z^{-4} + 0.1z^{-5}}{1 + 0.9z^{-2} + 0.2z^{-4}}$$

is a power-symmetric IIR transfer function.

Answer: $H(z) = \frac{1}{2}\left[A(z^2) + z^{-1}\right]$ where $A(z) = \dfrac{0.2 + 0.9z^{-1} + z^{-2}}{1 + 0.9z^{-1} + 0.2z^{-2}}$ is a stable allpass function.

Thus,

$$H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = \frac{1}{4}\left[A(z^2) + z^{-1}\right]\left[A(z^{-2}) + z\right] + \frac{1}{4}\left[A(z^2) - z^{-1}\right]\left[A(z^{-2}) - z\right] = 1.$$

Example E7.9: Show that the causal FIR transfer function $H(z) = \dfrac{1}{1+\alpha}(1 + \alpha z^{-1})$, $\alpha > 0$, is a BR function.

Answer: $\left|H(e^{j\omega})\right|^2 = \dfrac{1}{(1+\alpha)^2}\left\{(1 + \alpha\cos\omega)^2 + (\alpha\sin\omega)^2\right\} = \dfrac{1 + \alpha^2 + 2\alpha\cos\omega}{(1+\alpha)^2}$. Thus,

$\dfrac{d\left|H_1(e^{j\omega})\right|^2}{d\omega} = \dfrac{-2\alpha\sin\omega}{(1+\alpha)^2} < 0$, for $\alpha > 0$. The maximum value of $\left|H(e^{j\omega})\right| = 1$ at $\omega = 0$, and the minimum value is at $\omega = \pi$. On the other hand, if $\alpha < 0$, then $\dfrac{d\left|H(e^{j\omega})\right|^2}{d\omega} > 0$, In this case the maximum value of $\left|H(e^{j\omega})\right| = (1-\alpha)^2/(1+\alpha)^2 > 1$ at $\omega = \pi$, and the minimum value is at $\omega = 0$. Hence, $H(z)$ is BR only for $\alpha > 0$.

Example E7.10: Show that the causal stable IIR transfer function $H(z) = \dfrac{1-z^{-2}}{4+2z^{-1}+2z^{-2}}$ is a BR function.

Answer: $H(z) = \dfrac{1}{2}\left(1 - \dfrac{2+2z^{-1}+4z^{-2}}{4+2z^{-1}+2z^{-2}}\right)$, where $A_1(z) = \dfrac{2+2z^{-1}+4z^{-2}}{4+2z^{-1}+2z^{-2}}$ is a stable allpass function. Hence, $H(z)$ is BR.

Example E7.11: Show that the pair of causal stable IIR transfer functions $H(z) = \dfrac{-1+z^{-2}}{4+2z^{-1}+2z^{-2}}$ and $G(z) = \dfrac{3+2z^{-1}+3z^{-2}}{4+2z^{-1}+2z^{-2}}$ are doubly complementary.

Answer: Note $H(z)+G(z) = \dfrac{2+2z^{-1}+4z^{-2}}{4+2z^{-1}+2z^{-2}}$ implying that $H(z)$ and $G(z)$ are allpass complementary. Next, $H(z)H(z^{-1})+G(z)G(z^{-1})$

$$H(z)H(z^{-1})+G(z)G(z^{-1}) = \frac{-1+z^{-2}}{4+2z^{-1}+2z^{-2}}\,\frac{-1+z^{2}}{4+2z+2z^{2}} + \frac{3+2z^{-1}+3z^{-2}}{4+2z^{-1}+2z^{-2}}\,\frac{3+2z+3z^{2}}{4+2z+2z^{2}}$$

$$= \frac{1+1-z^{-2}-z^{2}+9+6z+9z^{2}+6z^{-1}+4+6z+9z^{-2}+6z^{-1}+9}{(4+2z^{-1}+2z^{-2})(4+2z+2z^{2})} = 1.$$ Hence, $H(z)$ and $G(z)$ are also power complementary. As a result, they are doubly-complementary.

Example E7.12: Determine the power-complementary transfer function of the BR transfer function $H(z) = \dfrac{2(1+z^{-1}+z^{-2})}{3+2z^{-1}+z^{-2}}$.

Answer: $H(z) = \dfrac{1}{2}\left(1 + \dfrac{1+2z^{-1}+3z^{-2}}{3+2z^{-1}+z^{-2}}\right)$, where $A(z) = \dfrac{1+2z^{-1}+3z^{-2}}{3+2z^{-1}+z^{-2}}$ is a stable allpass function. Hence, the power-complementary transfer function to $H(z)$ therefore is given by

$$G(z) = \frac{1}{2}\left(1 - \frac{1+2z^{-1}+3z^{-2}}{3+2z^{-1}+z^{-2}}\right) = \frac{1-z^{-2}}{3+2z^{-1}+z^{-2}}.$$

Example E7.13: Determine by inspection whether the second-order polynomial $D(z) = 1+0.92z^{-1}+0.1995z^{-2}$ has all roots inside the unit circle.

Answer: $|d_1| = 0.92$ and $1+d_2 = 1.1995$. Since $|d_1| < |1+d_2|$ and $|d_2| < 1$, both roots are inside the unit circle.

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Example E7.14: Test analytically the BIBO stability of the causal IIR transfer function $H(z) = \dfrac{z^2 + 0.3z - 99.17}{z^3 - \frac{1}{2}z^2 - \frac{1}{4}z + \frac{1}{12}}$.

Answer: $A_3(z) = \dfrac{\frac{1}{12} - \frac{1}{4}z^{-1} - \frac{1}{2}z^{-2} + z^{-3}}{1 - \frac{1}{2}z^{-1} - \frac{1}{4}z^{-2} + \frac{1}{12}z^{-3}}$. Note, $|k_3| = \frac{1}{12} < 1$. Using Eq. (7.148) we arrive at $A_2(z) = \dfrac{-0.2098 - 0.4825z^{-1} + z^{-2}}{1 - 0.4825z^{-1} - 0.2098z^{-2}}$. Here, $|k_2| = 0.2098 < 1$. Continuing this process, we get, $A_1(z) = \dfrac{-0.6106 + z^{-1}}{1 - 0.6106z^{-1}}$. Finally, $|k_1| = 0.6106 < 1$. Since $|k_i| < 1$, for $i = 3, 2, 1$, $H(z)$ is BIBO stable.
