CHAPTER

18 Beams of Composite Materials

LEARNING GOALS

After completing this chapter, you will be able to understand the following:

- Stress analysis of composite beam.
- What is a reinforced concrete beam?
- Stress analysis of reinforced concrete beam.
- Preliminary idea of the designing a reinforced concrete beam.
- What is balanced reinforced concrete beam?

In Chapter 6, we had discussed the normal or longitudinal stresses, also known as the bending stresses, which are developed due to the bending of a beam made up of a homogeneous material following Hooke's law. However, quite often in practical applications we have to come across beams that are made of two or more materials, called composite materials, which are no longer beams of homogeneous material. *Reinforced concrete beam* is an example where *the beam section is made up of concrete and steel rods*. To determine the stresses developed in such beams, we need to modify our foregoing beam theory as discussed in the following section.

18.1 Bending Stress in a Composite Beam

For the sake of simplicity, we consider a composite beam made up of two materials A and B and of rectangular cross-section with Young's moduli of elasticity as E_A and E_B ($E_B > E_A$), respectively. We show the beam section in Figure 18.1.

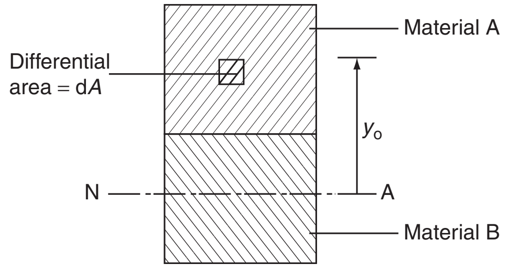

Figure 18.1 Cross-section of composite beam.

If we identify a differential area dA in material A at a distance of y from the neutral axis (NA) of the beam cross-section, then the force in the longitudinal direction acting on that area will be given by

$$\mathrm{d}F_{\mathrm{A}} = -\frac{E_{\mathrm{A}}\,y}{\rho}\,\mathrm{d}A \tag{18.1}$$

This is because dF_{A} is equal to $\sigma_{\mathrm{A}}\mathrm{d}A$ and σ_{A} equals to $E\in_{\mathrm{A}}$. Here $\in_{\mathrm{A}} = y/\rho$ where ρ is the radius of curvature of the neutral surface. In a similar way, the same area element dA in the material B will experience a force dF_{B} which is given by

$$\mathrm{d}F_{\mathrm{B}} = -E_{\mathrm{B}}\frac{y}{\rho}\,\mathrm{d}A \tag{18.2}$$

However, the same force equation can be written as

$$\begin{aligned}\mathrm{d}F_{\mathrm{B}} &= -\frac{E_{\mathrm{B}}}{E_{\mathrm{A}}}\frac{E_{\mathrm{A}}\,y}{\rho}\,\mathrm{d}A\\ &= \frac{E_{\mathrm{B}}}{E_{\mathrm{A}}}\left(-\frac{E_{\mathrm{A}}\,y}{\rho}\,\mathrm{d}A\right)\\ &= -E_{\mathrm{A}}\frac{y(n\mathrm{d}A)}{\rho}\end{aligned}$$

or

$$\mathrm{d}F_{\mathrm{B}} = -E_{\mathrm{A}}\frac{y\mathrm{d}A'}{\rho} \tag{18.3}$$

where

$$\mathrm{d}A' = n\mathrm{d}A = \frac{E_{\mathrm{B}}}{E_{\mathrm{A}}}\mathrm{d}A \quad \text{and} \quad n = \frac{E_{\mathrm{B}}}{E_{\mathrm{A}}}$$

Here, n is known as the *modular ratio*. If $E_{\mathrm{B}} > E_{\mathrm{A}}$ as assumed earlier, then n is greater than 1. If, however, $E_{\mathrm{B}} < E_{\mathrm{A}}$ then n is lower than 1.

Clearly Eqs. (18.1) and (18.3) indicate that if we transform the entire beam cross-section to material A, then the area of the material B is converted to n times its original area, where, we repeat, $n = E_{\mathrm{B}}/E_{\mathrm{A}}$. However, while transforming this area, we must keep y constant and hence, cross-sectional dimension of area of material B is changed *parallel* to the neutral axis of the section. Also, we must remember that *the neutral axis (NA) of the section shell passes through the centroid of the 'transformed' section*. Figure 18.2 shows the area transformation process of the beam section shown in Figure 18.1 earlier.

If the above beam is subjected to positive bending moment, then the stress distribution shown in Figure 18.3 is developed in the original beam cross-section and the *'transformed'* beam cross-section. In Figure 18.3(a), stress distribution in the original beam cross-section is shown, while in Figure 18.3(b), stress distribution in the transformed equivalent beam cross-section of material A is represented.

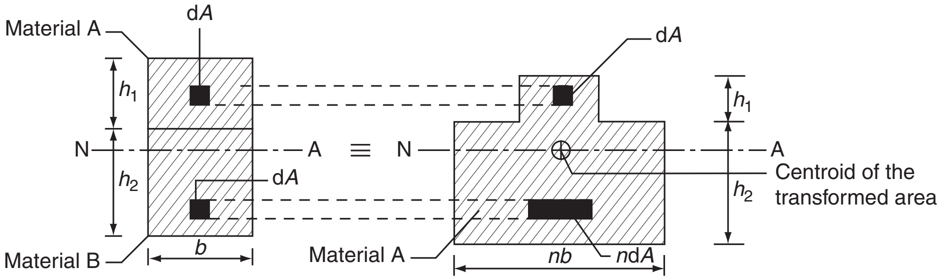

Figure 18.2 Area transformation.

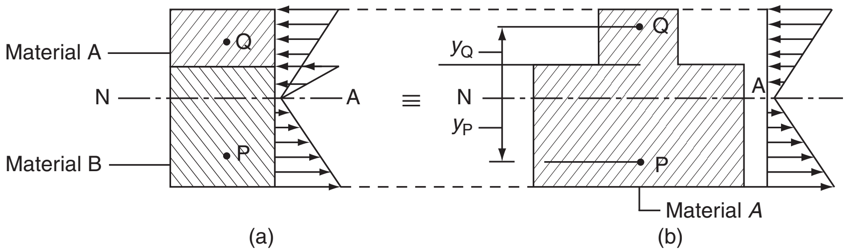

Figure 18.3 Stress distribution in beams with composite materials.

If we want to calculate the magnitude of stresses at points P and Q in materials B and A, respectively, we can determine them from the so-called 'transformed section' as

$$|\sigma_P| = E_A \frac{y_P}{\rho} \quad \text{and} \quad |\sigma_Q| = E_A \frac{y_Q}{\rho} \tag{18.4}$$

However, in reality, magnitude of stress developed at P should be given by

$$|\sigma_P|_{\text{true}} = E_B \frac{y_P}{\rho}$$

Multiplying and dividing by E_A, we get

$$|\sigma_P|_{\text{true}} = \frac{E_B}{E_A} E_A \frac{y_P}{\rho}$$

or

$$|\sigma_P|_{\text{true}} = n|\sigma_P| \tag{18.5}$$

where $|\sigma_P|$ denotes the magnitude of the stress at P in the transformed cross-section. Thus, to get the real stress at P (corresponding to the fact that it belongs to material B with modulus of elasticity E_B), we need to multiply the calculated stress by the modular ratio $n = E_B/E_A$, where E_A is the modulus of elasticity of the material in terms of which the entire area has been *transformed*.

EXAMPLE 18.1

Two metal strips are solidly bonded to form a metal bar of the cross-section as shown in Figure 18.4. With the given data, calculate the maximum bending moment that can be safely applied to the composite bar (refer Table 18.1).

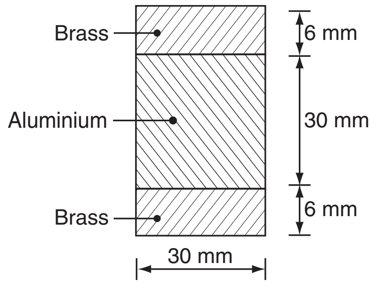

Figure 18.4 Beam cross-section.

Table 18.1 Material properties

Material	*E (GPa)*	*Allowable Stress (MPa)*
Brass	105	160
Aluminium	70	100

Solution

Let us transform the cross-section of the composite beam to the cross-section of aluminium beam as shown in Figure 18.5.

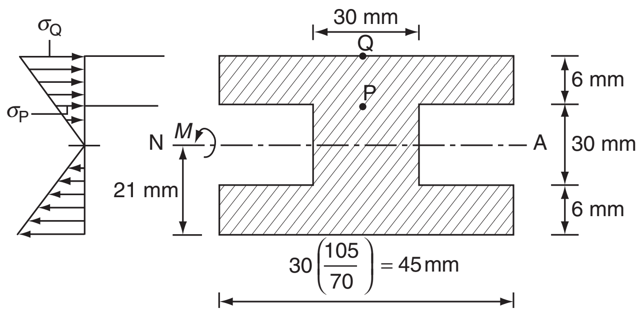

Figure 18.5 Transformed equivalent aluminium section.

Here, our modular ratio $n = E_B/E_A = 105/70 = 1.5$.
Therefore, width of brass section = 30(1.5) mm = 45 mm.

The section, as shown, is symmetric and the neutral axis passes half-way through it. Also, we identify two points P and Q belonging to the section representing farthest points in aluminium and brass sections, respectively.

The centroidal area moment of inertia is given by

$$\bar{I} = \left\{\frac{1}{12}(30)^4 + 2\left[\frac{1}{12}(45)(6)^3 + (45)(6)(18)^2\right]\right\} = 244080 \text{ mm}^4$$

Now, maximum stresses have been specified in the given data. We can find the transformed stress for the brass as 160/1.5 MPa [from Eq. (18.5)] = 106.67 MPa. Thus,

$$\sigma_{\text{P}} \le 100 \text{ MPa} \quad \text{and} \quad \sigma_{\text{Q}} \le 106.67 \text{ MPa}$$

It is known that due to bending moment M applied to the section, a linear stress distribution results as shown in the left half of the figure. Therefore,

$$\frac{\sigma_{\text{Q}}}{\sigma_{\text{P}}} = \frac{y_{\text{Q}}}{y_{\text{P}}} = \frac{21}{15} = 1.4$$

But

$$\left.\frac{\sigma_{\text{Q}}}{\sigma_{\text{P}}}\right|_{\text{allowable}} = \frac{106.67}{100} = 1.07 < 1.4$$

Clearly, we conclude from the above expression that we cannot consider $\sigma_{\text{P}} = \sigma_{\text{P}}|_{\text{allowable}}$ as σ_{Q} will be more than the specified limit. We set

$$\sigma_{\text{Q}} = \sigma_{\text{Q}}|_{\text{allowable}} = 106.67 \text{ MPa} \ \Rightarrow \ \sigma_{\text{P}} = \frac{106.67}{1.4} = 76.2 \text{ MPa}$$

which is less than $\sigma_{\text{P}}|_{\text{allowable}} = 100$ MPa. Thus, the safe maximum bending moment M to the section shall correspond to stress of 106.67 MPa at Q. Therefore,

$$\sigma_{\text{Q}} = \frac{MC}{I} \ \Rightarrow \ 106.67 = \frac{(M)(21)}{244080} \Rightarrow M = 1239810.17 \text{ Nm} = 1.24 \text{ kNm} \qquad \textbf{[Answer]}$$

EXAMPLE 18.2

A composite beam of aluminium strip (E_{Al} = 75 GPa) and a copper strip (E_{Cu} = 105 GPa) are bonded together as shown in Figure 18.6. If a bending moment M = 35 Nm is applied about a horizontal axis, calculate the maximum stresses developed in aluminium and copper strips.

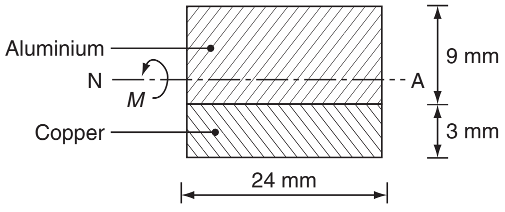

Figure 18.6 Beam section.

Solution

Let us transform the section into an equivalent aluminium section. The modular ratio is

$$n = \frac{E_{\mathrm{Cu}}}{E_{\mathrm{Al}}} = \frac{105}{75} = 1.4$$

so the new width of the copper section = (1.4)(24) mm = 33.6 mm.

Let us represent the transformed section in Figure 18.7:

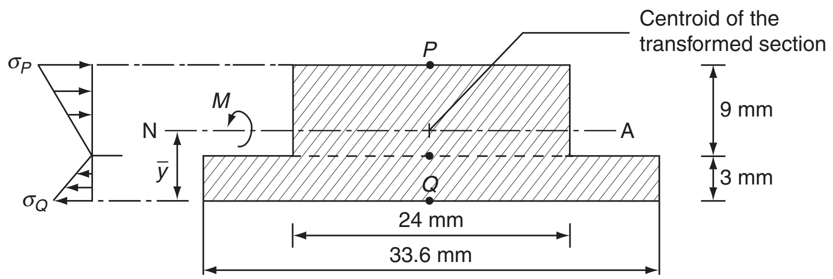

Figure 18.7 Transformed section.

Now we place the neutral axis (NA) of the section which shall pass through the centroid of the transformed equivalent section. Therefore, centroidal height by composite area method is

$$\bar{y} = \frac{(33.6)(3)(1.5) + (24)(9)(7.5)}{(33.6)(3) + (24)(9)} = 5.59 \text{ mm}$$

and centroidal area moment of inertia is

$$\begin{aligned}\bar{I} &= \frac{1}{12}(33.6)(3)^3 + (33.6)(3)(5.59+5)^2 + \frac{1}{12}(24)(9)^3 + (24)(9)(7.5-5.59)^2 \\ &= 4007.78 \text{ mm}^4\end{aligned}$$

In the figure, we show points P and Q belonging to the aluminium and copper sections, respectively, which are farthest from the neutral axis (NA) of the section. So we can calculate the magnitude of the maximum stresses at P and Q as

$$|\sigma_{\mathrm{P}}| = \sigma_{\max}\big|_{\mathrm{Al}} = \frac{35(10^3)(12-5.59)}{4007.78}\frac{\mathrm{N}}{\mathrm{mm}^2} = 55.98 \text{ MPa}$$

and $$|\sigma_{\mathrm{Q}}| = \sigma_{\max}\big|_{\mathrm{Cu}} = \frac{35(10^3)(5.59)}{4007.78} \times 1.4 \frac{\mathrm{N}}{\mathrm{mm}^2} = 68.34 \text{ MPa}$$

Assuming a positive bending moment applied to the section, we conclude that stress in aluminium will be compressive; and that in the copper strip it will be tensile in nature. Thus, maximum stress in the aluminium strip is 55.98 MPa (compressive) and maximum stress in the copper strip is 68.34 MPa (tensile) **[Answer]**

EXAMPLE 18.3

Let us consider a composite beam of two materials (1) and (2) having elastic moduli E_1 and E_2 with $E_2 < E_1$ as shown in Figure 18.8. The beam has a circular cross-section of diameter d. Calculate the position of the centroid of the transformed section which is equivalent to material (1).

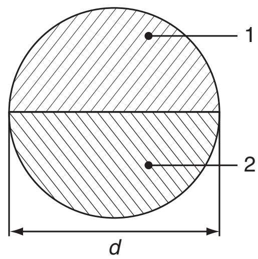

Figure 18.8 Beam section.

Solution

From the given condition, the transformed section is equivalent to material (1) and thus, material (2) will become semi-elliptic as shown in Figure 18.9:

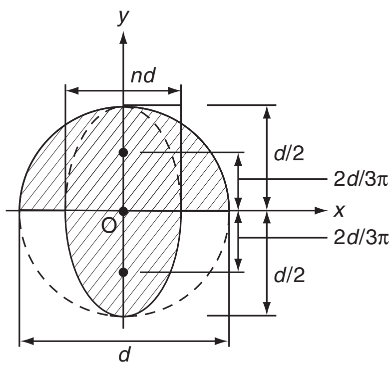

Figure 18.9 Transformed section.

Clearly, the width of semicircle of material (2) is

$$\text{Width} = \frac{E_2}{E_1} d = nd$$

where $n < 1$. Now the location of the centroid of the semi-elliptical section, with semi-major axis $a = d/2$ and semi-minor axis $b = nd/2$, is determined as follows:

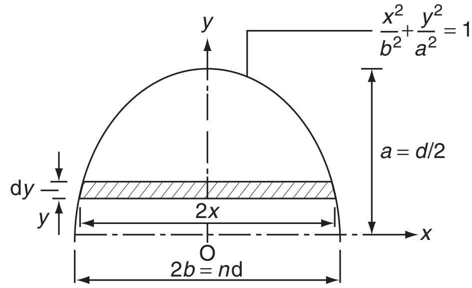

Figure 18.10 Centroid of semi-ellipse.

From Figure 18.10, we get

$$\bar{y} = \frac{\int y\,\mathrm{d}A}{\int \mathrm{d}A} = \frac{\int_0^a y(2x)\mathrm{d}y}{\pi ab/2} = \frac{2}{\pi(d/2)(nd/2)} \int_0^{a=d/2} 2y\frac{b}{a}\sqrt{a^2-y^2}\;\mathrm{d}y$$

or

$$\bar{y} = \frac{16}{n\pi d^2}\int_0^a ny\sqrt{a^2-y^2}\;\mathrm{d}y$$

$$= -\frac{8}{\pi d^2}\int_0^a \sqrt{a^2-y^2}\,\mathrm{d}(a^2-y^2) \quad [\text{as } \mathrm{d}(a^2-y^2) = -2y\,\mathrm{d}y]$$

$$= -\frac{8}{\pi d^2}\frac{2}{3}\Big[(a^2-y^2)^{3/2}\Big]_0^a = \frac{16}{3\pi d^2}a^3$$

or

$$\bar{y} = \frac{16}{3\pi d^2}\frac{d^3}{8} = \frac{2d}{3\pi}$$

The locations of the centroid as measured from the x-axis for the semicircular area and semi-elliptical area are shown in Figure 18.9. Now, applying composite area theorem to Figure 18.10, to determine the centroid of the entire transformed area, we get

$$\bar{y} = \frac{\left(\frac{\pi d^2}{8}\right)\left(\frac{2d}{3\pi}\right)+\left(\frac{n\pi d^2}{8}\right)\left(-\frac{2d}{3\pi}\right)}{\frac{\pi d^2}{8}+\frac{n\pi d^2}{8}} = \frac{2d}{3\pi}\frac{(1-n)}{(1+n)}$$

Thus, the centroid is at a distance of $(2d/3\pi)[(1-n)/(1+n)]$ above from the horizontal diameter.

[Answer]

▶ **Note:** For the transformed section in the above example, we can also determine the centroidal area moment of inertia. It is first determined for the semi-ellipse about the x-axis as

$$I_{xx} = \iint_A y^2\,\mathrm{d}A = \int_0^a y^2\,2x\,\mathrm{d}y = \left(\frac{2b}{a}\right)\int_0^a y^2\sqrt{a^2 - y^2}\,\mathrm{d}y$$

Putting $y = a\sin\theta$, we get

$$I_{xx} = 2na^4 \int_0^{\pi/2} \sin^2\theta\cos^2\theta\,\mathrm{d}\theta$$

or

$$I_{xx} = \left(\frac{na^4}{2}\right)\int_0^{\pi/2} \sin^2 2\theta\,\mathrm{d}\theta = \frac{na^4}{4}\int_0^{\pi/2}(1-\cos 4\theta)\,\mathrm{d}\theta = \frac{n\pi a^4}{8} = \frac{n\pi a^4}{128}$$

Now by parallel axis theorem, we get

$$\bar{I}_{xx} = I_{xx} - A\bar{y}^2 = \frac{n\pi d^4}{128} - \left(\frac{n\pi d^2}{8}\right)\left(\frac{2d}{3\pi}\right)^2$$

$$= \frac{n\pi d^4}{128} - \frac{nd^4}{18\pi} = nd^4\left(\frac{\pi}{128} - \frac{1}{18\pi}\right)$$

Similarly, for the semicircular portion:

$$I_{xx} = \frac{\pi d^4}{128} \quad \text{and} \quad \bar{I}_{xx} = \frac{\pi d^4}{128} - \frac{\pi d^2}{8}\left(\frac{2d}{3\pi}\right)^2 = d^4\left(\frac{\pi}{128} - \frac{1}{18\pi}\right)$$

The centroidal area moment of inertia for the whole section is

$$\bar{I}_{xx}\Big|_{\text{whole}} = nd^4\left(\frac{\pi}{128} - \frac{1}{18\pi}\right) + \frac{n\pi d^2}{8}\left(\frac{2d}{3\pi}\,\frac{1-n}{1+n} + \frac{2d}{3\pi}\right)^2$$

$$+d^4\left(\frac{\pi}{128} - \frac{1}{18\pi}\right) + \frac{\pi d^2}{8}\left(\frac{2d}{3\pi} - \frac{2d}{3\pi}\,\frac{1-n}{1+n}\right)^2$$

$$= \left(\frac{\pi}{128} - \frac{1}{18\pi}\right)d^4(1+n) + \frac{n\pi d^2}{8}\cdot\frac{4d^2}{9\pi^2}\;\frac{4}{(1+n)^2} + \frac{\pi d^2}{8}\cdot\frac{4d^2}{9\pi^2}\,\frac{4n^2}{(1+n)^2}$$

$$= (n+1)d^4\left(\frac{\pi}{128} - \frac{1}{18\pi}\right) + \frac{2d^4}{9\pi}\,\frac{n}{(1+n)^2} + \frac{2d^4}{9\pi}\,\frac{n^2}{(1+n)^2}$$

Therefore,

$$\bar{I}_{xx}\Big|_{\text{whole}} = (n+1)d^4\left(\frac{\pi}{128} - \frac{1}{18\pi}\right) + \frac{2d^4}{9\pi}\,\frac{n}{(1+n)}$$

The above expression will be required to calculate stresses in such sections.

EXAMPLE 18.4

A rectangular beam is made of a material which has modulus of elasticity E_t in tension and E_c in compression (e.g., plastics are examples of these materials which have unequal moduli of elasticity). Writing the flexure equation as

$$\frac{1}{\rho} = \frac{M}{E_{\text{eq}}\bar{I}}$$

where $\bar{I} = bh^3/12$ (b is the width and h is the depth of the section). Prove that the equivalent modulus of elasticity E_{eq} is given by

$$E_{\text{eq}} = \frac{4E_{\text{t}}E_{\text{c}}}{\left(\sqrt{E_{\text{t}}} + \sqrt{E_{\text{c}}}\right)^2}$$

Solution

Let us assume that the neutral axis (NA) of the section divides the cross-section shown in Figure 18.11:

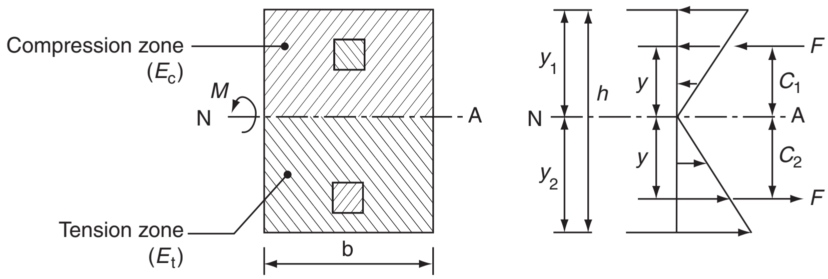

Figure 18.11 Beam section.

Let us assume that NA divides the section in the ratio y_1:y_2. Therefore, the stresses at points at a distance y above and below the NA are:

$$\sigma_{\text{comp}} = E_{\text{c}}\frac{y}{\rho} \quad \text{and} \quad \sigma_{\text{tensile}} = E_{\text{t}}\frac{y}{\rho}$$

where ρ is the radius of curvature of the neutral surface. If we build up an area dA surrounding these points, we get the resultant compressive and tensile forces, which are equal in magnitude.

$$F = \int_0^{y_1} \sigma_{\text{comp}}\ \mathrm{d}A = \int_0^{y_2} \sigma_{\text{tensile}}\ \mathrm{d}A$$

or

$$\int_0^{y_1} \frac{E_{\text{c}}yb\,\mathrm{d}y}{\rho} = \int_0^{y_2} \frac{E_{\text{t}}yb\,\mathrm{d}y}{\rho} \quad \Rightarrow \quad E_{\text{c}}y_1^2 = E_{\text{t}}y_2^2$$

Therefore,

$$\frac{y_1}{y_2} = \frac{\sqrt{E_{\text{t}}}}{\sqrt{E_{\text{c}}}} \Rightarrow y_1 = \frac{\sqrt{E_{\text{t}}}}{\sqrt{E_{\text{t}}} + \sqrt{E_{\text{c}}}}h \quad \text{and} \quad y_2 = \frac{\sqrt{E_{\text{c}}}}{\sqrt{E_{\text{t}}} + \sqrt{E_{\text{c}}}}h \qquad \textbf{(1)}$$

Now, taking moments of the forces about the NA, we get

$$M = \int_0^{y_1} E_c \frac{y^2}{\rho} b \, dy + \int_0^{y_2} E_t \frac{y^2}{\rho} b \, dy$$

$$= \frac{b}{3\rho}(E_c y_1^3 + E_t y_2^3)$$

Putting y_1 and y_2 from Eq. (1), we get

$$M = \frac{bh^3}{3\rho}\left(E_c \frac{E_t}{\sqrt{E_t} + \sqrt{E_c}} \sqrt{E_t} + E_t \frac{E_c}{\sqrt{E_t} + E_c} \sqrt{E_c} \right) \frac{1}{\left(\sqrt{E_t} + \sqrt{E_c}\right)^2}$$

or
$$M = \frac{bh^3}{3\rho} \frac{E_t E_c}{\left(\sqrt{E_t} + \sqrt{E_c}\right)^2} = \left(\frac{bh^3}{12}\right) \frac{1}{\rho} \frac{4E_t E_c}{\left(\sqrt{E_t} + \sqrt{E_c}\right)^2}$$

$$= \frac{\bar{I} E_{eq}}{\rho}$$

where $\bar{I} = bh^3/12$. Rearranging, we get

$$\frac{1}{\rho} = \frac{M}{E_{eq} \bar{I}}$$

where

$$E_{eq} = \frac{4E_t E_c}{\left(\sqrt{E_t} + \sqrt{E_c}\right)^2}$$

[Hence proved]

18.2 Reinforced Concrete Beam

We have already mentioned that reinforced concrete beam is an example of a beam comprising two different materials. It has great practical applications in the field of structural engineering[1]. Though the complete discussion on such beams is out of the scope of this book; however, what follows is an elementary beam model for the purpose of an elementary introduction to such important structures. Before we progress further, we make some important assumptions:

1. Concrete is a substance which is weak in tension and strong in compression. Hence, the entire compressive load on the beam section is borne by the concrete, while the steel rod takes up the entire tensile force acting on the beam.

[1]An interested reader can go through any standard textbook on the theory of reinforced concrete structures; refer to Reference (1) in the Bibliography section of the book.

2. Concrete follows Hooke's law approximately[2] (although it is not quite correct).
3. There is enough cohesive force between the concrete and the steel rods.

Essentially, a reinforced concrete beam is made up of concrete and steel rods that are inserted into it from its bottom since the bottom portion is subjected to tensile force as shown in Figure 18.12.

The steel rods are covered by a distance of 40–50 mm to protect them from being exposed to any possible fire hazard. Section depth must be assumed to be h. From the foregoing assumption, it is customary to ignore the concrete portion below NA as it does not carry any tensile force, which is developed in the beam section due to the application of positive bending moment M as shown.

We now efficiently transform the section into an equivalent concrete section, wherein the total steel rod area, $A_{\text{steel}} = N \times (\pi d^2/4)$, ($N$ and d being the number and diameter of the steel rod, respectively) is transformed into area $= nA_{\text{steel}}$. Since Young's modulus of steel ($E_{\text{steel}} = 200$ GPa) is more than that of the concrete ($E_{\text{conc}} = 25$ GPa), steel area is magnified by $n = E_{\text{steel}}/E_{\text{conc}}$ 8 times as shown Figure 18.12(b).

It is an important task to position the NA in the section defined by the parameter λ ($0 < \lambda < 1$) as can be seen from Figure 18.12(b) itself. Since the compressive force developed in the concrete section is equal to the total tensile force developed in the transformed steel section, we get

$$F = \int_0^{\lambda h} \frac{E_{\text{conc}}\, yb\mathrm{d}y}{\rho} = \int_0^{(1-\lambda)h} \frac{E_{\text{conc}}\, y}{\rho}\,\mathrm{d}A = \frac{E_{\text{conc}}}{\rho}(1-\lambda)h \cdot \left(\frac{E_{\text{steel}}}{E_{\text{conc}}}\right) A_{\text{S}}$$

or

$$\frac{E_{\text{conc}}\, b}{2\rho} \cdot \lambda^2 h^2 = \frac{E_{\text{conc}}}{\rho}\left(\frac{E_{\text{steel}}}{E_{\text{conc}}}\right)(1-\lambda)h\; A_{\text{S}}$$

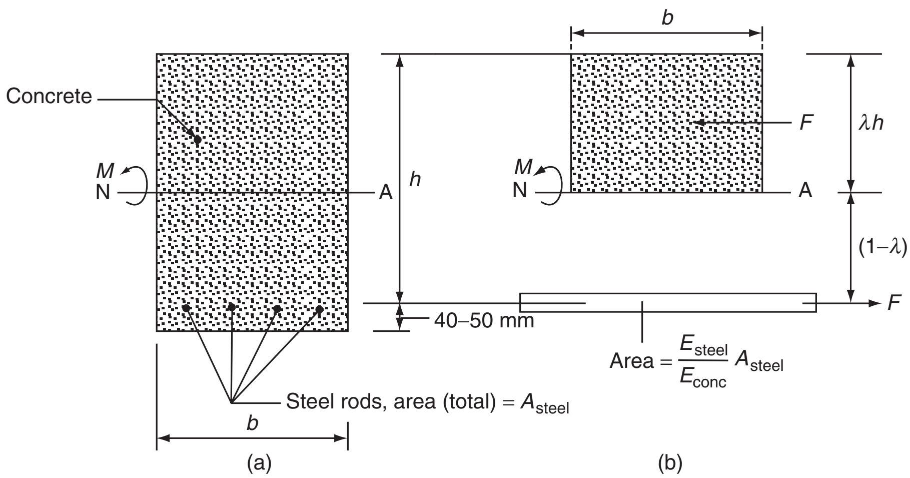

Figure 18.12 (a) Reinforced concrete beam section, (b) equivalent concrete section.

[2]Refer to Reference (12) in the Bibliography section of the book.

$$\left(\frac{bh}{2}\right)\lambda^2 = n(1-\lambda)A_{\text{steel}} \tag{18.6}$$

where $n = E_{\text{steel}}/E_{\text{conc}}$, which is the modular ratio. The above equation can be solved for λ to obtain the position of NA. To determine the second area moment of inertia, we write

$$\bar{I} = \frac{1}{3}b(\lambda h)^3 + (1-\lambda)^2 h^2 \cdot nA_{\text{steel}} \tag{18.7}$$

Clearly, stresses in the concrete and steel rods due to M can be determined as

$$\sigma_{\text{conc}} = \frac{M(\lambda h)}{\bar{I}}$$

and

$$\sigma_{\text{steel}} = n\frac{M(1-\lambda)h}{\bar{I}} \tag{18.8}$$

For a given bending moment M, the stresses calculated as per the above equations must be less than or equal to the allowable stresses in the concrete and steel rods. Conversely, if stresses are specified, the moment can be calculated using the above equations; and the maximum bending moment that can be applied to the reinforced concrete beam is given by

$$M_{\max} = \text{Min}\left(\frac{\sigma_{\text{conc}} \cdot \bar{I}}{\sigma h}, \frac{\sigma_{\text{steel}}\bar{I}}{n(1-\lambda)h}\right) \tag{18.9}$$

EXAMPLE 18.5

For the reinforced concrete beam shown in Figure 18.13, find the maximum bending moment M that can safely be applied to the section. Assume the modular ratio, n to be 15 for the grade of concrete used, the allowable stress of steel, σ_{steel} = 125 MPa and that of concrete, σ_{conc} = 4.5 MPa.

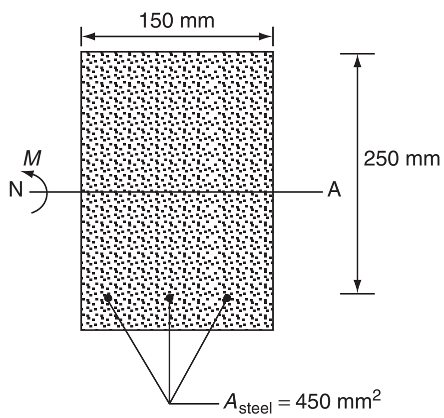

Figure 18.13 Reinforced beam section of Example 18.5.

Solution

The neutral axis (NA) position locator, λ can be found from the Eq. (18.6):

$$\left(\frac{bh}{2}\right)\lambda^2 = nA_{\text{steel}}(1-\lambda) \Rightarrow \lambda^2 = \left(\frac{2nA_{\text{steel}}}{bh}\right)(1-\lambda)$$

Putting the given values in the above equation, we get

$$\lambda^2 = \frac{2\times 15\times 450}{(150)(250)}(1-\lambda) = 0.36(1-\lambda)$$

Solving, we get

$$\lambda = \frac{-0.36 \pm \sqrt{0.36^2 + 4(0.36)}}{2} = 0.446$$

(ignoring the negative impossible root)

From Eq. (18.7),

$$\bar{I} = \frac{1}{3}b(\lambda h)^3 + (1-\lambda)^2 h^2 n\, A_S$$

$$= \frac{1}{3}(150)(0.446\times 250)^3 + (1-0.446)^2(250)^2(15)(450)$$

or $\bar{I} = 198.79\times 10^6\ \text{mm}^4$

Now from Eq. (18.9), we get

$$M_{\max} = \text{Min}\left(\frac{\sigma_{\text{conc}},\ \bar{I}}{\lambda h}, \frac{\sigma_{\text{steel}}\bar{I}}{n(1-\lambda)h}\right)$$

$$= \text{Min}\left(\frac{4.5\times 198\cdot 19(10^9)}{0.446\times 250}, \frac{125\times 198\cdot 19(10^9)}{15(1-0.446)\times 250}\right)$$

$$= \text{Min}(8022914.8\ \text{N mm},\ 11960890.5\ \text{N mm})$$

$$= 8022914.8\ \text{N mm} = 8.02\ \text{kN m}$$

Thus, the safe moment that can be applied to the section is $M_{\max} = 8.05$ kN m. **[Answer]**

EXAMPLE 18.6

The reinforced concrete beam shown in Figure 18.14 is subjected to the bending moment $M = +175$ kN m. If E_{steel} = 200 GPa and E_{conc} = 25 GPa, calculate the maximum stresses developed in the steel and concrete rods.

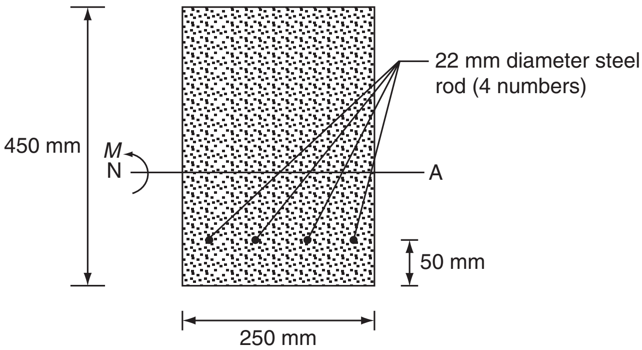

Figure 18.14 Reinforced concrete beam section.

Solution

For the given problems, $A_{\text{steel}} = 4\times(\pi/4)(22)^2\ \text{mm}^2 = 1520.5\ \text{mm}^2$ and the modular ratio, $n = E_{\text{steel}}/E_{\text{conc}} = 200/25 = 8.0$. Clearly, the neutral axis (NA) locator λ in the equivalent concrete section is given by the Eq. (18.6) as

$$\lambda^2 = \left(\frac{2nA_{\text{steel}}}{bh}\right)(1-\lambda) = \frac{2\times 8\times 1520.5}{250\times 400}(1-\lambda) = 0.243(1-\lambda)$$

(note that $h = 450 - 50 = 400$ mm). Therefore,

$$\lambda = \frac{1}{2}\left[-0.243 + \sqrt{0.243^2 + 4(0.24)}\right] = 0.386$$

Now, from Eq. (18.7)

$$\bar{I} = \frac{1}{3}b(\lambda b)^3 + (1-\lambda)^2 h^2 (nA_{\text{steel}})$$

$$= \frac{1}{3}(250)(0.386\times 400)^3 + (1-0.386)^2 \times 400^2 \times (8\times 1520.5)$$

or $$\bar{I} = 1.0405\times 10^9\ \text{mm}^4$$

By using stress equations from Eq. (18.8), we get

$$\sigma_{\text{conc}} = \frac{M(\lambda h)}{\bar{I}}$$

or $$\sigma_{\text{conc}} = \frac{175(10^6)(0.386 \times 400)}{1.0405(10^9)} \text{ N/mm}^2 = 25.97 \text{ MPa}$$

and $$\sigma_{\text{steel}} = n\frac{M(1-\lambda)h}{\bar{I}} = 8.0\frac{175(10^6)(1-0.386)(400)}{1.0405(10^9)} \text{ N/mm}^2$$

or $$\sigma_{\text{steel}} = 330.5 \text{ MPa}$$

Thus, the maximum stresses developed in the concrete is 26.0 MPa (compressive) and 330.5 MPa (tensile). **[Answer]**

▶ **Note:** We observe from the stress equations in Eq. (18.8) that the ratio of the maximum stress in the steel to that in concrete is

$$r = \frac{\sigma_{\text{steel}}}{\sigma_{\text{conc}}} = n\frac{1-\lambda}{\lambda} = n\left(\frac{1}{\lambda} - 1\right)$$

and from Eq. (18.6), we note that:

$$\lambda = \frac{1}{2}\left[\sqrt{k^2 + 4k} - k\right]; \quad k = \frac{2nA_{\text{steel}}}{bh}$$

$$\frac{1}{\lambda} = \frac{2}{\sqrt{k^2+4k} - k} = \frac{1}{2k}\left[\sqrt{k^2+4k} + k\right]$$

or $$\left(\frac{1}{\lambda} - 1\right) = \frac{\sqrt{k^2+4k}}{2k} + \frac{1}{2} - 1 = \frac{\sqrt{k^2+4k}}{2k} - \frac{1}{2} = \frac{1}{2k}\left[\sqrt{k^2+4k} - k\right] = \frac{\lambda}{k}$$

So the maximum stress ratio is

$$r = \frac{\sigma_{\text{steel}}}{\sigma_{\text{conc}}} = \frac{n\lambda}{k}$$

Sometimes, beams are designed in such a way that the maximum stresses in the concrete and steel rods are equal to their specified allowable values. These beams are called *balanced reinforced concrete* beams. In order to design a reinforced concrete beam as a balanced one, we need to position the neutral axis (NA) locator, λ. If the specified stresses are $(\sigma_{\text{conc}})_{\text{allowable}}$ and $(\sigma_{\text{steel}})_{\text{allowable}}$ for concrete and steel, respectively, then λ can be found as:

$$n\left(\frac{1}{\lambda} - 1\right) = \frac{(\sigma_{\text{steel}})_{\text{allowable}}}{(\sigma_{\text{conc}})_{\text{allowable}}}$$

or $$\frac{1}{\lambda} = 1 + \frac{(\sigma_{\text{steel}})_{\text{allowable}}}{(\sigma_{\text{conc}})_{\text{allowable}}}\frac{1}{n} = 1 + \frac{(\sigma_{\text{steel}})_{\text{allowable}}}{(\sigma_{\text{conc}})_{\text{allowable}}} \cdot \frac{E_{\text{conc}}}{E_{\text{steel}}}$$

$$\lambda = \frac{1}{1 + \dfrac{(\sigma_{\text{steel}})_{\text{allowable}}}{(\sigma_{\text{concrete}})_{\text{allowable}}} \cdot \dfrac{E_{\text{concrete}}}{E_{\text{steel}}}} \tag{18.10}$$

The above expression locates the *NA in balanced beam*.

EXAMPLE 18.7

A reinforced concrete beam is designed to carry a bending moment, M = 10.4 kN. The width b and depth h of the section are related as h = 1.5 b. Assume $\sigma_{steel} = 125$ *MPa,* $\sigma_{conc} = 4.0$ *MPa,* $E_{steel} = 210$ *GPa and* $E_{conc} = 14$ *GPa. Design the beam.*

Solution

We first determine the locator of NA using the specified stress ratios from Eq. (18.10) as:

$$\lambda = \frac{1}{1 + \dfrac{(\sigma_{\text{steel}})_{\text{allowable}}}{(\sigma_{\text{conc}})_{\text{allowable}}} \cdot \dfrac{E_{\text{conc}}}{E_{\text{steel}}}} = \frac{1}{1 + \dfrac{125}{4.0} \cdot \dfrac{14}{210}} = 0.324 \tag{1}$$

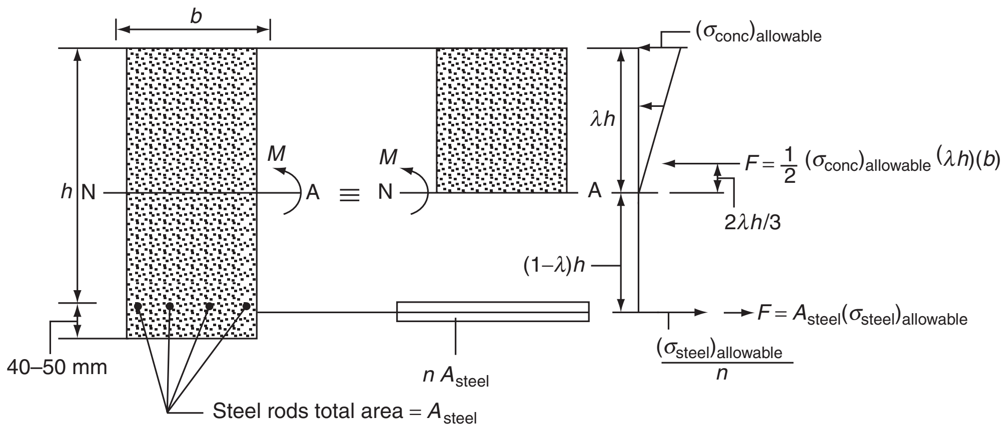

Figure 18.15 Concrete beam.

Now from the equivalent diagram shown in Figure 18.15 using the stress distribution, we get

$$\frac{1}{2}(\sigma_{\text{conc}})_{\text{allowable}}\,\lambda\, bh = A_{\text{steel}}(\sigma_{\text{steel}})_{\text{allowable}}$$

$$\frac{bh}{A_{\text{steel}}} = 2\frac{(\sigma_{\text{steel}})_{\text{allowable}}}{\lambda(\sigma_{\text{conc}})_{\text{allowable}}} = 2\frac{125}{(0.324)(4.0)} = 192.9 \tag{2}$$

Again from the stress distribution of the equivalent concrete section, by equating the internal moment with the applied bending moment, we get:

$$F\left(\frac{2\lambda h}{3}+h-\lambda h\right)=M$$

or

$$Fh\left(1-\frac{2\lambda}{3}\right)=M$$

But $F = A_{\text{steel}}(\sigma_{\text{steel}})_{\text{allowable}}$. So, the above equation becomes:

$$A_{\text{steel}}h\left(1-\frac{2\lambda}{3}\right)(\sigma_{\text{steel}})_{\text{allowable}} = M$$

Putting λ from Eq. (1), we get

$$(A_{\text{steel}})(h)=\frac{M}{\left(1-\frac{2\lambda}{3}\right)(\sigma_{\text{steel}})_{\text{allowable}}}=\frac{10.4(10^6)}{\left(1-\frac{2\times 0.324}{3}\right)(125)}$$

$$(A_{\text{steel}})(h)=93273.54\ \text{mm}^2 \tag{3}$$

Now from Eqs. (2) and (3), we get

$$bh^2=(192.9)(93273.54)$$

as it is known that $h = 1.5b$ or $b = h/1.5$, we get

$$h^3=(1.5)(192.9)(93273.54) \quad \Rightarrow \quad h = 299.96\ \text{mm} \simeq 300\ \text{mm}$$

Thus, section width, $b = 300/1.5 = 200$ mm and the steel area is

$$A_{\text{steel}} = 93273.54/h = 93273.54/300 = 310.9\ \text{mm}^2$$

Using 10 mm steel rods, we have number of steel rods = $N = 310.9/[(\pi/4)(10)^2] = 3.96 \approx 4$. Thus we design the beam as shown in Figure 18.16.

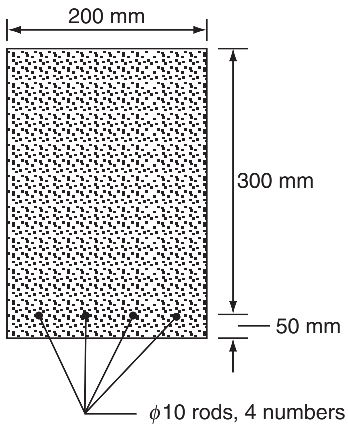

Figure 18.16 Designed beam cross-section.

[Answer]

Summary

In this chapter, we have elaborated our earlier knowledge of stress equation of beam, which was developed for a *homogenous* beam material, to the cases where different materials are used to form composite beams. To this end, we have introduced the concept of equivalent section and successfully applied stress equation for composite beams.

We have also given a careful attention to a special case of composite beam of practical importance, known as *reinforced concrete beams*.

Some design aspects for such beams are also discussed in order to introduce the reader to a specialised topic on the theory of concrete structures.

Key Terms

Normal stress
Longitudinal stress
Bending stress
Composite area method
Homogeneous material
Neutral axis
Transformed section
Modular ratio
Reinforced concrete beam
Balanced reinforced concrete beam

Review Questions

1. What do you mean by beams of composite materials?
2. Explain the transformed section of a composite beam.
3. What do you mean by reinforced concrete beam?
4. What are the assumptions of stress analysis of a reinforced concrete beam?
5. What do you understand by balanced reinforced concrete beam?

Numerical Problems

1. A simply supported beam with 3 m span carries a concentrated load $P = 10$ acting at its midpoint. The beam is made of wood and steel and has cross-section as shown in Figure 18.17. Assuming modular ratio $E_{wood}/E_{steel} = 0.05$, calculate the stresses developed in steel and wood.

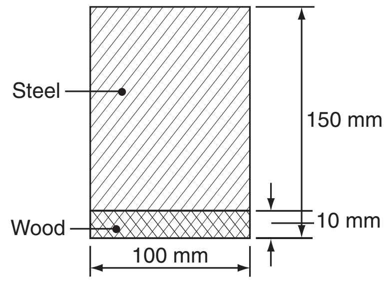

Figure 18.17 Problem 1.

2. A composite beam has a circular cross-section as shown in Figure 18.18. If E_{brass} = 103 GPa and E_{Al} = 70 GPa and moment M = 904 N m is applied to the section, calculate the maximum stresses in brass and aluminium.

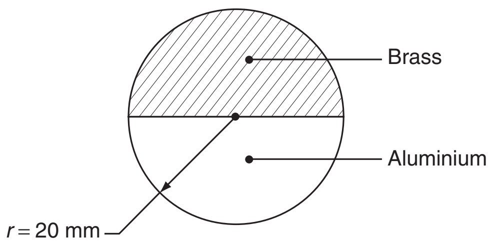

Figure 18.18 Problem 2.

[*Hint*: Use the results of Example 18.3.]

3. For the reinforced concrete beam shown in Figure 18.19, the bending moment applied is M = +203 kN m. Calculate the stresses in concrete and steel. Assume E_{conc} = 25 GPa, E_{steel} = 200 GPa.

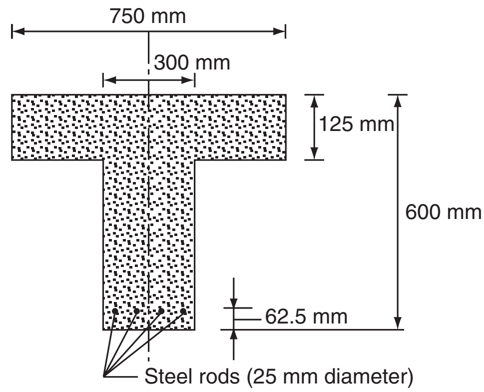

Figure 18.19 Problem 3.

4. For the concrete shown in Figure 18.20, E_{conc} = 16.67 GPa and E_{steel} = 200 GPa. Find M_{max} if $(\sigma_{conc})_{allowable}$ = 12 MPa and $(\sigma_{steel})_{allowable}$ = 110 MPa.

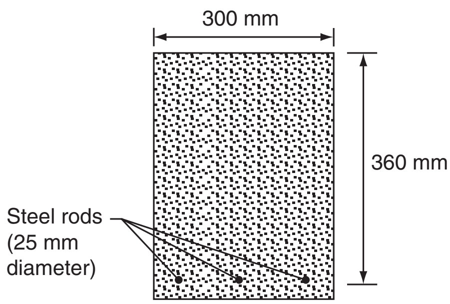

Figure 18.20 Problem 4.

Answers

Numerical Problems

1. σ_{steel} = 77.5 MPa (tensile); σ_{wood} = 12.4 MPa (compressive)
2. σ_{brass} = 155 MPa (compressive); σ_{steel} = 123 MPa (tensile)
3. σ_{steel} = 200 MPa (tensile), σ_{conc} = 8.0 MPa (compressive)
4. M_{max} = 50 kN m