CHAPTER

17 Leaf Springs

LEARNING GOALS

After completing this chapter, you will be able to understand the following:

- Beam shape of uniform strength.
- What is leaf spring and where it is used?
- Stress and deformation analyses of leaf spring.
- Designing of a leaf spring in terms of determination of the leaf's requisite cross-sectional dimension.

Leaf springs (sometimes also known as *laminated springs*) are made up of beams of *uniform strength* and are normally used in automobiles to absorb the shock and vibration produced by road undulations, thereby providing comfort to passengers. Figure 17.1 shows a general arrangement of a typical *semi-elliptical* leaf spring.

The figure also shows the different components of a typical leaf spring[1]. Figure 17.2 shows a typical application of the spring in an automobile.

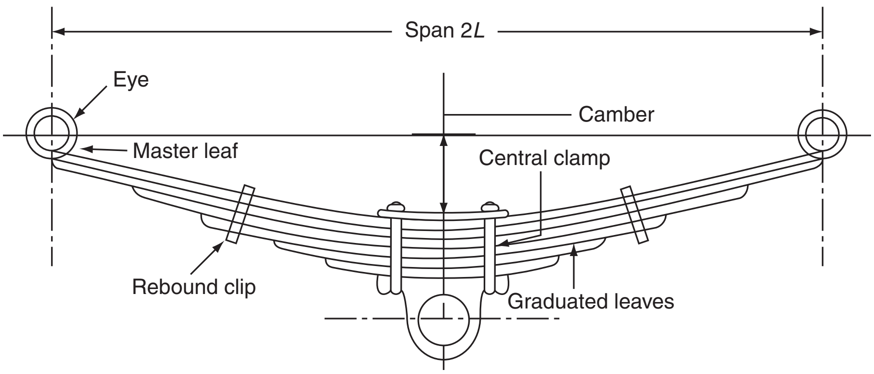

Figure 17.1 Laminated semi-elliptical spring.

[1] An interested reader is further directed to refer to Reference (24) in the Bibliography section of the book to know further details of a leaf spring.

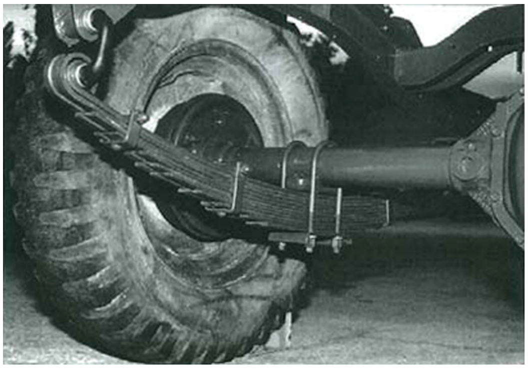

Figure 17.2 Leaf spring in an automobile.

17.1 Beams of Uniform Strength

Let us consider a cantilever beam of a rectangular cross-section loaded at its free-end as shown in Figure 17.3. Now, we know that for such beams, the bending stress is same throughout the length of the beam.

If we consider a section at a distance of x from the free-end of the beam as shown in Figure 17.3(a) and show its free-body diagram in Figure 17.3(b), we can find out the magnitude of the bending moment M_x prevailing at that section according to the equation:

$$M_x = -Px \tag{17.1}$$

Obviously, the magnitude of the maximum bending stress is given by:

$$\sigma = \frac{MC}{I} = \frac{(Px)\left(\frac{t}{2}\right)}{\frac{1}{12}bt^3}$$

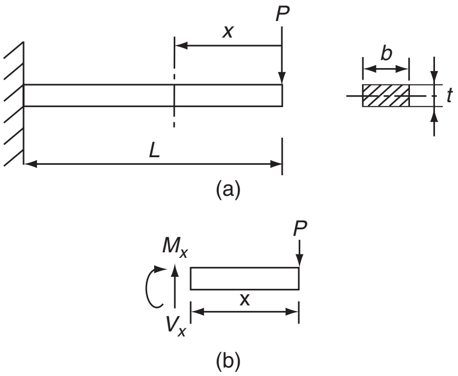

Figure 17.3 (a) Cantilever beam, (b) free-body diagram of any section.

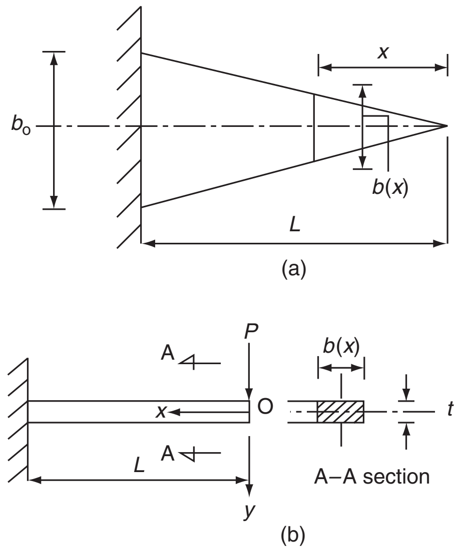

Figure 17.4 Beam of uniform strength: (a) Top view of beam, (b) front view of beam.

or
$$\sigma = \frac{6Px}{bt^2} \tag{17.2}$$

If we keep the thickness t of the beam section constant and vary the section width b along the beam length in such a way that everywhere the stress is same, say equal to σ_o, then Eq. (17.2) can be used to define b as:

$$b = \left(\frac{6P}{\sigma_o t^2}\right)x$$

or
$$b = \lambda x \tag{17.3}$$

For a given value of P, the parameter $\lambda = 6P/\sigma_o t^2$ is constant and evidently the above Eq. (17.3) suggests that b must then vary linearly with x as shown in the Figure 17.4(a).

However, practically, absence of any material at the cantilever tip as shown in Figure 17.4(a) is unacceptable as load P (at the tip of the beam) then cannot be borne by it. This above design of beam is the basic element of a leaf spring.

17.2 Deflection of Beam of Uniform Strength

As is evident from the previous Figure 17.4(a), the width of the section at a distance x from the free-end of the beam is given by:

$$b(x) = \left(\frac{b_o}{L}\right)x$$

and centroidal area moment of inertia is given by:

$$I(x) = \frac{1}{12} b(x) t^3$$

$$= \frac{1}{12}\left(\frac{b_0 t^3}{L}\right) x$$

or $$I(x) = \left(\frac{I_0}{L}\right) x \tag{17.4}$$

To compute the bending deflection y, we use the flexure equation, Eq. (7.6) of Chapter 7 as:

$$EI \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -M_x$$

Although in Chapter 7, we mostly used the above equation for a *prismatic beam* (i.e., beam with constant EI), now we use it for a non-prismatic beam as well. From Eq. (17.1), as $M_x = -Px$, we get

$$\frac{EI_o}{L} x \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = Px$$

or $$\left(\frac{EI_o}{L}\right) \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = P$$

Integrating successively with respect to x:

$$\left(\frac{EI_o}{L}\right) \frac{\mathrm{d}y}{\mathrm{d}x} = Px + C_1 \tag{17.5}$$

and $$\left(\frac{EI_o}{L}\right) y = \frac{Px^2}{2} + C_1 x + C_2 \tag{17.6}$$

Putting the boundary conditions of $y = 0$ and $\mathrm{d}y/\mathrm{d}x = 0$ at $x = L$ in the above equations, we obtain:

$$C_1 = -PL \quad \text{and} \quad C_2 = \frac{PL^2}{2}$$

Thus, putting $x = 0$ in Eq. (17.6) we get the maximum deflection, which is the deflection of the beam at its free end as:

$$\left(\frac{EI_o}{L}\right) \delta_{\max} = \frac{Px^2}{2} - PLx + \frac{PL^2}{2}\bigg|_{x=0}$$

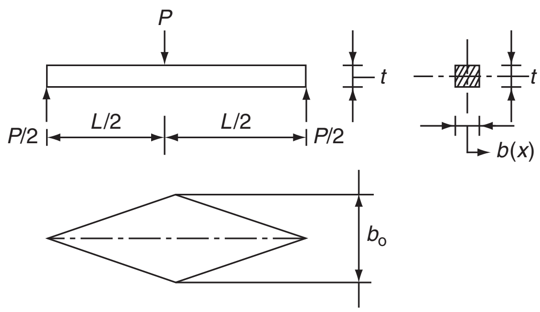

Figure 17.5 Lozenge-shaped beam.

or
$$\frac{EI_o}{L}\delta_{max} = \frac{PL^2}{2}$$

or
$$\delta_{max} = \frac{PL^3}{2EI_o} \tag{17.7}$$

Comparing the free-end deflection of a prismatic cantilever beam with area moment of inertia equal to I_o, we note that the above deflection is 1.5 times more. This large deflection under load P is used to make the beam to act as a spring. If instead of a cantilever beam had we considered a simply supported beam, the uniform strength would have led to the geometry known as *Lozenge-shaped* geometry as shown in Figure 17.5.

As each half of the above beam can be modelled as a cantilever beam of uniform strength, we get from Eqs. (17.2) and (17.5) by putting $L/2$ in place of L and $P/2$ in place of P, the following relations for the lozenge-shaped beam:

$$\begin{aligned}\sigma_o &= \frac{3PL}{b_o t^2}\\ \delta_{max} &= \frac{PL^3}{3LEI_o}\end{aligned} \tag{17.8}$$

17.3 Leaf Spring

A leaf spring or a laminated spring is made up from a beam of uniform strength by cutting, say n_g equal strips from the original beam of uniform strength and stacking them one on top of the other. The construction is shown in Figure 17.6.

This arrangement of stacked plates produces *cantilever-type leaf spring* as shown in Figure 17.6(b). In Figure 17.6(a), a triangular plate of width b_o (which is a cantilever of uniform strength) is cut into n_g equal strips of width b of the central strip (1) where $b_o = n_g b$ and stacked on as shown in Figure 17.6(b)

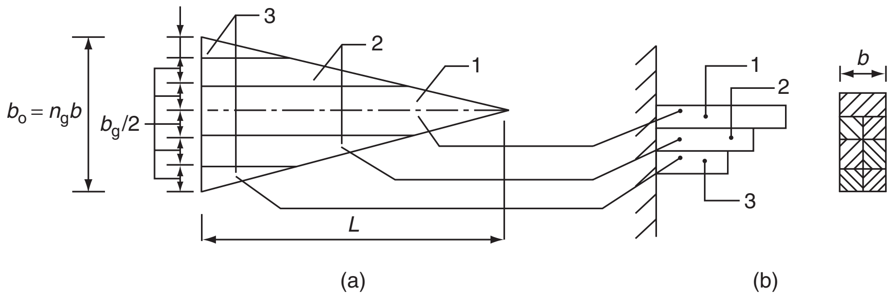

Figure 17.6 Cantilever leaf spring.

to form a cantilever-type leaf spring. In practice, one or more number of extra full-length plates of uniform width are used to place on top of the graduated plate (1). This is done because to carry the load at the tip of plate (1) we need sufficient material to provide the necessary shear force. However, as plate (1) has a pointed tip, it is not possible to do so if no extra top plate(s) is/are used. This extra plate is known as the *master leaf*.

Yet another variation of the laminated spring, known as the *semi-elliptic spring* as shown in Figure 17.1. It is used in practice, where a favourable curvature known as *cambering* is provided to the entire assembly of plates to carry more loads with uniform stress distribution using favourable conditions of residual stresses. The load on the beam due to which this initial curvature can be reduced to zero is called *proof load*.

In the following sections, we provide the analysis of the cantilever-type leaf springs, which is applicable also for the semi-elliptic leaf springs.

Stress Deformation Analysis for Leaf Springs

Let us assume that to the top of the assembly of the n_g number of graduated plates of width b, we use n_m number of extra full-length master leaf each of uniform width b [i.e., width of the topmost graduated plate (1)]. Thus, essentially, we have a cantilever beam of width $n_m b$ and length L connected *parallel* to the other cantilever beam of uniform strength of width $n_g b$ and length L as shown in the Figure 17.7. Clearly, through this arrangement, the applied load P is shared by both the beams. The parallel connectivity of the two beams is modelled as if the beams are connected by a massless rigid link connected at the free ends of the beams. Let P_m and P_g be the loads shared by the master leaf and the graduated leaf, respectively. Clearly,

$$P = P_m + P_g \tag{17.9}$$

Also, we note that the free-end deflections of the beams are equal as they are connected by a rigid link. Hence,

$$\frac{P_m L^3}{3EI_m} = \frac{P_g L^3}{2EI_g}$$

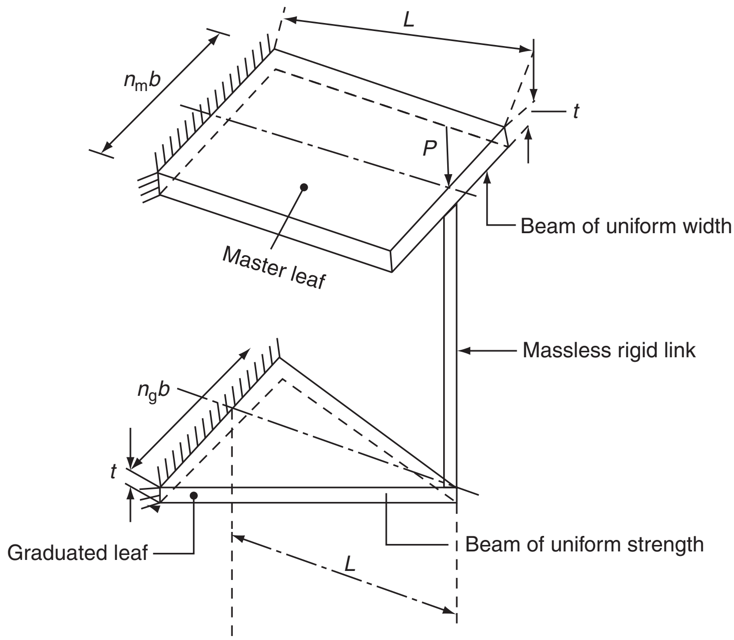

Figure 17.7 Parallel assembly of cantilever leaf springs.

or

$$\frac{P_m}{P_g} = \frac{3}{2} \cdot \frac{I_m}{I_g}$$

where I_m and I_g are second moments of area of the master leaf and graduated leaf, respectively. Therefore,

$$\frac{P_m}{P_g} = \frac{3}{2} \frac{\left(\frac{1}{12} n_m b t^3\right)}{\left(\frac{1}{12} n_g b t^3\right)}$$

or

$$\frac{P_m}{P_g} = \frac{3}{2} \frac{n_m}{n_g} \tag{17.10}$$

Solving Eqs. (17.9) and (17.10), we get

$$P_m = \frac{3n_m}{3n_m + 2n_g} P = \text{Load carried by master leaf} \tag{17.11}$$

and
$$P_g = \frac{2n_g}{3n_m + 2n_g} P = \text{Load carried by the graduated leaf} \tag{17.12}$$

Now, maximum stress developed in the beams' master and graduated leaves are as follows:

$$\sigma_m = \frac{6P_m L}{n_m bt^2}$$

in the master leaf as it is a rectangular section. Putting P_m from Eq. (17.11), we get

$$\sigma_m = \frac{18}{(3n_m + 2n_g)} \frac{PL}{bt^2} \tag{17.13}$$

The above equation gives us the maximum stress developed in the master leaf. Similarly, by putting the expression of P_g in the stress equation, we get the maximum stress in the graduated leaf as

$$\sigma_g = \frac{12}{3n_m + 2n_g} \frac{PL}{bt^2} \tag{17.14}$$

Comparing Eqs. (17.13) and (17.14), we observe that the stress developed in the master leaf is 50% more than that of the laminated plates. Obviously, this limits the load-carrying capacity of the entire assembly. Hence, in practice, the master leaf is given a different radius of curvature, thereby putting favourable residual stress in it in such a way that when the load is applied to the assembly, an equal stress distribution exists and the system can carry more load. Let us now focus on the deflection of the entire assembly. If δ_{max} is the maximum free-end deflection of the assembly, then from Eq. (17.7):

$$\delta_{max} = \frac{P_g L^3}{2EI_g}$$

Now from Eq. (17.12) and noting $I_g = n_g bt^3/12$, we get

$$\delta_{max} = \frac{6L^3}{n_g bt^3 E} \frac{2n_g}{3n_m + 2n_g} P$$

or
$$\delta_{max} = \frac{12PL^3}{Ebt^3(3n_m + 2n_g)} \tag{17.15}$$

Equations (17.13) and (17.14) indicate the strength of the spring and Eq. (17.15) gives the rigidity of the assembly. We have to note that the above modelling is approximate, and hence the foregoing equations approximately depict the strength and deformation analysis of the cantilever-type of laminated springs. Recall that these equations are also applicable to the semi-elliptic springs.

EXAMPLE 17.1

A cantilever leaf spring is designed to meet the following specifications:

Load on the spring = 2 kN
Total number of leaves = 8
Number of extra full-length leaves = 2
Width of each leaf = 50 mm
Length of spring = 500 mm
Design stress in tension = 350 MPa

What is the thickness of leaf required to meet the above requirements?

Solution

Assuming no pre-stressing, we observe that stress in the master leaf is the deciding factor as its stress is 50% more than that in the graduated leaves. From Eq. (17.13), we get

$$\sigma_{\mathrm{m}} = \frac{18PL}{bt^2}\frac{1}{3n_{\mathrm{m}} + 2n_{\mathrm{g}}}$$

So, the thickness of the leaf is given by

or $$t = \sqrt{\frac{18PL}{\sigma_{\mathrm{m}} b}\frac{1}{3n_{\mathrm{m}} + 2n_{\mathrm{g}}}}$$

Putting the necessary values, we get

$$t = \sqrt{\frac{(18)(2)(10^3)(0.5)}{(350)(10^6)(0.05)}\cdot\frac{1}{3(2)+2(8)}} \times 10^{-3}\ \text{mm}$$
$$= 6.84\ \text{mm}$$

Thus, the required thickness of the plates is 6.84 mm. **[Answer]**

EXAMPLE 17.2

A laminated semi-elliptic spring under a central load of 12 kN is to have an effective length of 1 m and is not allowed to deflect more than 75 mm. The spring has 10 leaves, 2 of which are of full length and are pre-stressed so that all leaves have the same stress after the full load is applied. All leaves have the same width and thickness. The maximum stress in the leaves is not to exceed 350 MPa. Find the width and thickness of the plates.

Solution

In the given condition, the stresses are all equal in the leaves. We thus, consider the semi-elliptic spring as two cantilever-type leaf springs connected as shown in Figure 17.8.

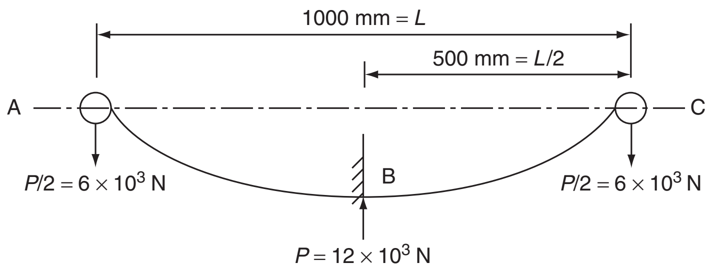

Figure 17.8 Arrangement of beams in Example 17.2.

As shown in the figure, we consider the portion BC of the beam as the cantilever-type leaf spring with load $P/2$ and length $L/2$. Applying the stress and deflection equations, and by noting that stresses are all equal in the leaves, we get

$$\sigma_{\max} = \frac{6M}{nbt^2} = \frac{6\left(\frac{PL}{4}\right)}{nbt^2} = \frac{3}{2}\frac{PL}{nbt^2}$$

where n is the total number of leaves, b denotes plate width and t is plate thickness. (Note that the above equation is applied as the stresses in the leaves are all equal because of pre-stressing.)

$$\frac{3}{2}\frac{12(10^3)(10^3)}{(10)bt^2} = 350 \Rightarrow bt^2 = \frac{3(12)(10^6)}{2(10)(350)}$$

or $$bt^2 = 5142.86 \text{ mm}^3 \qquad \textbf{(1)}$$

Again, in Eq. (17.15) by putting $P/2$ and $L/2$ in place of P and L, respectively, we get

$$\delta_{\max} = \frac{12\left(\frac{P}{2}\right)\left(\frac{L}{2}\right)^3}{Ebt^3(3n_m + 2n_g)}$$

$$= \frac{3}{4}\frac{PL^3}{Ebt^3(3n_m + 2n_g)}$$

or $$bt^3 = \frac{3}{4}\frac{PL^3}{E\delta_{\max}(3n_m + 2n_g)}$$

$$= \frac{3}{4}\frac{12(10^3)(1000)^3}{200(10^3)(75)(3\times 2 + 2\times 8)}$$

or $$bt^3 = 27272.73 \text{ mm}^4 \quad (2)$$

Now from Eqs. (1) and (2), we get

$$b = 183.1 \text{ mm} \quad \text{and} \quad t = 5.30 \text{ mm}$$

Thus, the required width and thickness of the plates are 183.1 mm and 5.30 mm, respectively. **[Answer]**

EXAMPLE 17.3

A 100 mm outer diameter steel coil spring having 10 active coils of 2.5 mm diameter wire is in contact with a 750 mm long steel cantilever spring having 6 graduated leaves, 100 mm wide and 6.5 mm thick as shown in Figure 17.9. Assume for steel, E = 200 GPa.

(a) Calculate force F, which when gradually applied to the top of the coil spring will cause the cantilever spring to deflect by 25 mm.

(b) What will be the maximum shear stress in the coil spring?

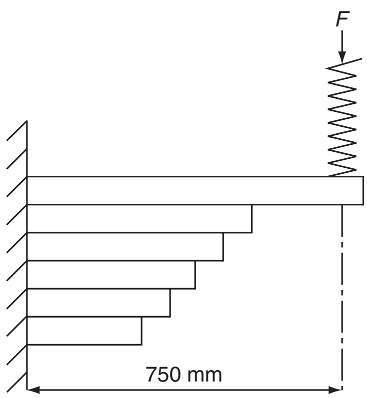

Figure 17.9 Problem 17.3.

Solution

We observe from the figure that the coiled spring and the graduated leaf spring are in series arrangement and carry the same axial load *F*.

(a) By applying deflection equation, Eq. (17.15) for the leaf spring, we get

$$\delta_{\max} = \frac{12FL^3}{Ebt^3(3n_{\mathrm{m}} + 2n_{\mathrm{g}})}$$

or $$F = \frac{(Ebt^3)\delta_{\max}(3n_{\mathrm{m}} + 2n_{\mathrm{g}})}{12L^3}$$

Now putting the given values in the above expression, we get

$$F = \frac{200(10^3)(100)(6.5)^3(25)(3\times 0 + 2\times 6)}{12(750)^3}\,\text{N}$$

or $$F = 325.5\ \text{N} \qquad \textbf{[Answer]}$$

(b) Recalling our Eqs. (2.15) and (2.16) from Section 2.5, we get the following results. The maximum shear stress in the coiled spring considering Wahl's correction factor is

$$\tau_{\max} = \frac{8FD}{\pi d^3}\left[\frac{4C-1}{4C-4} + \frac{0.615}{C}\right]$$

Putting $C = D/d = 100/12.5 = 8$

$$\tau_{\max} = \frac{(8)(325.5)(100)}{\pi(12.5)^3}\left[\frac{4(8)-1}{4(8)-4} + \frac{0.615}{8}\right]\frac{\text{N}}{\text{mm}^2}$$

or $$\tau_{\max} = 50.25\ \text{MPa} \qquad \textbf{[Answer]}$$

and considering theoretical stress equation, we get

$$\tau_{\max} = \frac{8FD}{\pi d^3}\left[1 + \frac{d}{2D}\right]$$

$$= \frac{8FD}{\pi d^3}\left[1 + \frac{0.5}{C}\right]$$

$$= \frac{(8)(325.5)(100)}{\pi(12.5)^3}\left[1 + \frac{0.5}{8}\right]\frac{\text{N}}{\text{mm}^2}$$

or $$\tau_{\max} = 45.1\ \text{MPa} \qquad \textbf{[Answer]}$$

Summary

In this chapter, we have carefully introduced the concept of beam of *uniform strength*. From that concept, we went on discussing the use of such beams with constant thickness as spring elements, as they suffer more deflection than their ordinary counterparts. These designs (two of which

have been considered – cantilever type, simply supported type) find their useful practical applications especially in the case of automobiles.

A preliminary approximate stress analysis of these springs, known as leaf springs, have been presented along with their deformation characteristics also. This chapter will surely be able to introduce the students to a complex design analysis of leaf springs, which they will take up in their Machine Design course.

Key Terms

Leaf spring	Cantilever beam	Graduated leaf
Laminated spring	Prismatic beam	Coiled spring
Uniform strength beam	Lozenge-shaped beam	Wahl's correction factor
Semi-elliptical leaf spring	Cantilever-type leaf spring	
Rectangular cross-section	Master leaf	

Review Questions

1. What is a leaf spring?
2. Explain the function of a leaf spring.
3. Why is a leaf spring used?
4. What is proof load?
5. What do you mean by beam of uniform strength?
6. Discuss the construction of a leaf spring.
7. Derive the stress deformation theory of leaf springs.

Numerical Problems

1. A 1 m long cantilever spring is composed of 8 graduated leaves and 1 additional full-length leaf. The leaves are 45 mm wide. A load of 2000 N at the free-end of the spring causes a deflection of 75 mm. Assuming no pre-stressing, calculate the maximum bending stress developed in the spring.
2. Repeat the above problem assuming pre-stressing of the additional full-length leaf.
3. For Problem 1, calculate the thickness of the leaf required.

Answers

Numerical Problems

1. 277 MPa
2. 195 MPa
3. 12.3 mm