CHAPTER

19 Statically Indeterminate Beams – Continuous Beams

LEARNING GOALS

After completing this chapter, you will be able to understand the following:

- What is a statically determinate beam?
- Calculation of the degree of statical indeterminacy of such beams.
- What is a continuous beam?
- Complete analysis of a continuous beam.
- Approach towards taking up more advanced studies in structural mechanics.

As we already know that for any structure if statical equations are not sufficient to determine the external support reactions for given external loading, it is called *statically indeterminate*. Additional equations defining the deformation characteristics of such members are required in order to find the external reactions. For example, for the two-dimensional bending problems when we reduce our problem of bending to plane-bending situations, only *three independent equations of equilibrium* ($\sum F_x = 0$, $\sum F_y = 0$, and $\sum M_z = 0$, if bending occurs in *x*-*y* plane) are available. As beams mostly carry transverse loading, say in *y*-direction, the first equilibrium equation, that is, $\sum F_x = 0$, becomes trivial and as such, we have only two independent equations of equilibrium (i.e., $\sum F_y = 0$ and $\sum M_z = 0$). Clearly we can only solve two unknown reactions, and the beam configurations shown in Figure 19.1 are called *statically determinate beams*, as equations of the equilibrium are sufficient to determine their support reactions completely.

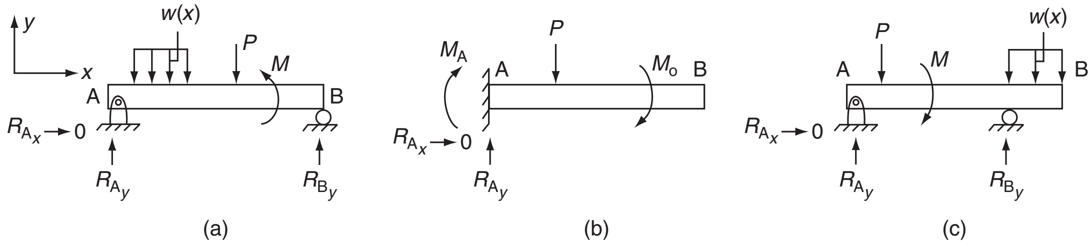


Figure 19.1 Statically determinate beams.

On the other hand, if we increase the number of supports for the above beams, they reduce to *statically indeterminate ones* as shown in the Figure 19.2.

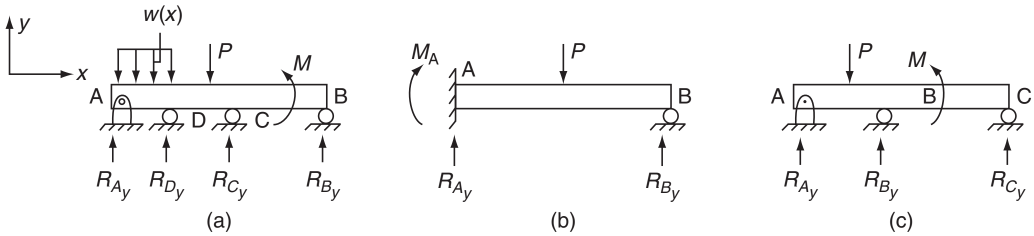


Figure 19.2 Statically indeterminate beams.

In each of the above beam configurations, we can define the term called *degree of indeterminacy* as the number of unknown support reactions are less than the number of equilibrium equations (in this case, 2). Hence for cases shown in Figures 19.2(a)–(c), the degree of indeterminacy is 2, 1 and 1, respectively. We have already shown in Chapter 10, how to deal with statically indeterminate beams with energy approach. Here, we present another method. Whenever we have a number of roller supports and one hinged support for a beam, we call such beams as *continuous beams.* Accordingly, beams shown in Figure 19.2 can also be called continuous beams. In order to solve for the support reactions of continuous beams, we can follow the method as outlined in the next section.

19.1 Analysis of Continuous Beams

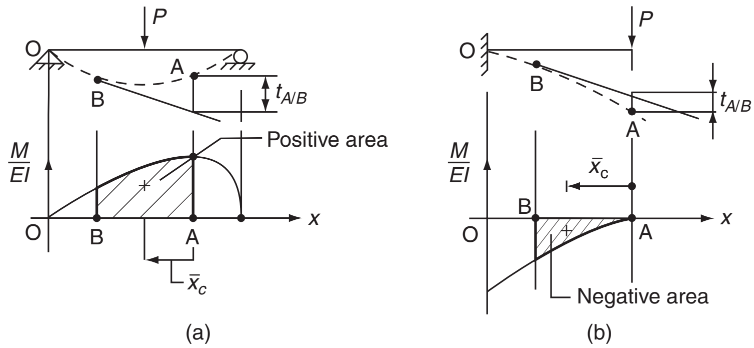
The solution methodology for such beams is based on the *principle of superposition* which is always true for a *linear* structure. As long as our beam materials follow Hooke's law and the associated deformations are small enough, we can expect linear behaviour and principle of superposition can be well-justified.

The basic concept of the analysis of the continuous beams relies on the *second area-moment theorem* which we have already discussed in Chapter 7; but for the sake of convenience, we reiterate it as exemplified below:

Second Area-Moment Theorem

The tangential deviation (i.e., distance of a point on the elastic curve from the tangent drawn to the elastic curve at another point) of point A with respect to the tangent drawn at B is $t_{A/B}$ (where both points A and B are on the elastic curve) is equal to the first area moment of M/EI vs. x (distance measured along the beam) between points A and B about a transverse axis drawn through point A.

If A is vertically above the tangent drawn at B, then $t_{A/B}$ is considered positive, otherwise it will be negative. Figures 19.3(a) and (b) explain the theorem for two cases (a) and (b) of beam loading shown therein.



Area of M/EI vs. x diagram is *positive* between points A and B and $t_{A/B} > 0$ $t_{A/B} = \bar{x}_C \times$(Area of M/EI *vs*. x-curve between A and B)	Area of M/EI vs. x diagram is negative between points A and B and $t_{A/B} < 0$ $t_{A/B} = \bar{x}_C \times$(Area of M/EI *vs*. x-curve between A and B)

Figure 19.3 Second area-moment theorem.

19.2 Three-Moment Equation

Let us now consider the following continuous beam shown in Figure 19.4(a) where at least one support is pinned or hinged and the remaining ones can be considered as roller supports.

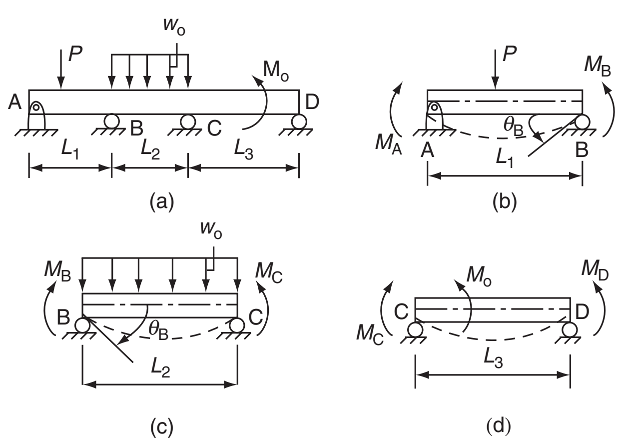


Figure 19.4 Continuous beam.

The continuous beam is segmented into several simply supported beams as shown in Figures 19.4(b)–(d), where the unknown bending moments at terminal points are shown. It is amply clear that *in order to analyse such continuous beams, we need to determine the support reactions completely and to this end, we must know these bending moments.*

If, for example, we consider the beam segments AB and BC and denote them as nth and $(n + 1)$th member of a long continuous beam and also denote the terminal ends A, B and C as $(n - 1)$, n and $(n + 1)$ points, then we note that the relationship of the angles θ_B at the nth and $(n + 1)$th member from Figures 19.4(b) and (d) is

$$\theta_B\big|_{n\text{th member}} = -\theta_B\big|_{(n+1)\text{th member}}$$

or

$$\theta_B\big|_{n\text{th member}} + \theta_B\big|_{(n+1)\text{th member}} = 0 \tag{19.1}$$

The above is the key equation for analysis of a continuous beam. The equation stems from the basic material continuity of the beam segments. Now, θ_B (i.e., slope of the tangent drawn to the elastic curve at point B) can be determined from Figure 19.5.

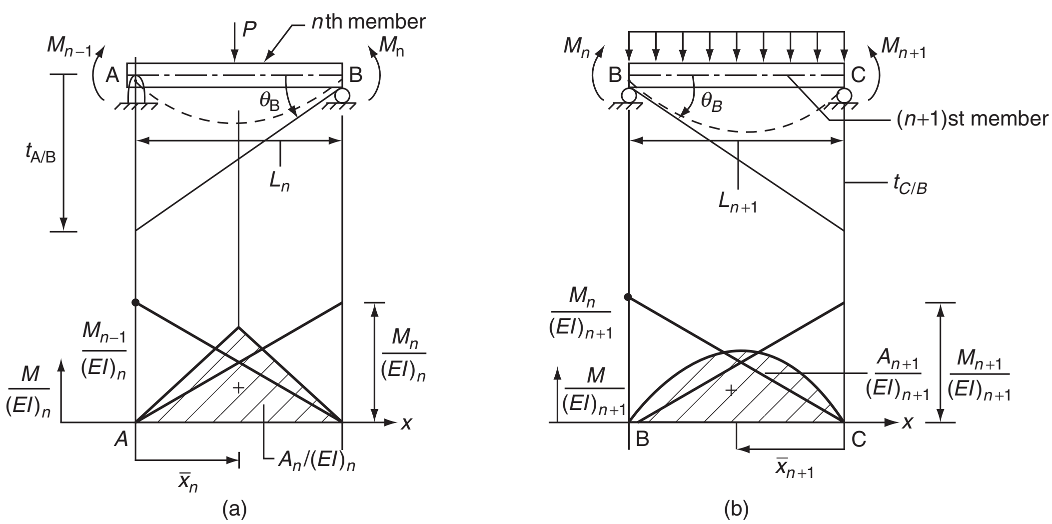


Figure 19.5 *M/EI* diagrams of beam segments.

As shown in the figure, we consider the *M/EI* diagrams for the nth and $(n + 1)$th beam members where the *moment area for the external load acting on these segments is shown hatched*. While in the figures, the end-point moment diagrams are also shown. To represent the most general case, we consider the flexural rigidity, *EI*, to be different for these two segments. From Figure 19.5(a), we note that for the nth member, the first moment of *M/EI vs. x* area with respect to point A is

$$\tan\theta_B\big|_{n\text{th member}} \cong \theta_B\big|_{n\text{th member}} = \frac{t_{A/B}}{L_n} = \frac{1}{L_n} \times \begin{Bmatrix} \text{First moment of } M/EI \\ \text{vs. } x \text{ area with respect} \\ \text{to point A} \end{Bmatrix}$$

$$\theta_B\big|_{n\text{th member}} = \frac{1}{L_n}\left\{\frac{A_n}{(EI)_n}\cdot\bar{x}_n + \frac{1}{2}\frac{M_{n-1}}{(EI)_n}L_n\cdot\frac{L_n}{3} + \frac{1}{2}\frac{M_n}{(EI)_n}L_n\frac{2L_n}{3}\right\}$$

or

$$\theta_B\big|_{n\text{th member}} = \frac{A_n\bar{x}_n}{L_n(EI)_n} + \frac{M_{n-1}}{(EI)_n}\frac{L_n}{6} + \frac{M_n}{(EI)_n}\frac{L_n}{3}$$

or

$$\theta_B\big|_{n\text{th member}} = \frac{1}{(EI)_n}\left[\frac{A_n\bar{x}_n}{L_n} + \frac{M_{n-1}L_n}{6} + \frac{M_nL_n}{3}\right] \tag{19.2}$$

Similarly, from Figure 19.5(b), we note for the $(n+1)$th member:

$$\tan\theta_B\big|_{(n+1)\text{th member}} \cong \theta_B\big|_{(n+1)\text{th member}} = \frac{t_{C/B}}{L_{n+1}}$$

Therefore,

$$\theta_B\big|_{(n+1)\text{th member}} = \frac{1}{(EI)_{n+1}}\left[\frac{A_{n+1}\bar{x}_{n+1}}{L_{n+1}} + \frac{M_nL_{n+1}}{3} + \frac{M_{n+1}L_{n+1}}{6}\right] \tag{19.3}$$

From Eqs. (19.1)–(19.3), we get:

$$(EI)_{n+1}\left[\frac{A_n\bar{x}_n}{L_n} + \frac{M_{n-1}L_n}{6} + \frac{M_nL_n}{3}\right] + (EI)_n\left[\frac{A_{n+1}\bar{x}_{n+1}}{L_{n+1}} + \frac{M_nL_{n+1}}{3} + \frac{M_{n+1}L_{n+1}}{6}\right] = 0$$

or

$$\begin{aligned}&(EI)_{n+1}M_{n-1}L_n + 2\left\{(EI)_{n+1}L_n + (EI)_nL_{n+1}\right\}M_n + (EI)_nM_{n+1}L_{n+1}\\ &= -6\left\{(EI)_{n+1}\frac{A_n\bar{x}_n}{L_n} + (EI)_n\frac{A_{n+1}\bar{x}_{n+1}}{L_{n+1}}\right\}\end{aligned} \tag{19.4}$$

The above equation, sometimes called *three-moment equation*, is repeatedly used for each segment of a continuous beam. We thus get a set of *simultaneous equations* of the *unknown bending moments.* The above equation is due to Clapeyron[1]. The moments in the equation, in turn, can be used to determine the support reactions completely and further analysis can easily be done. Equation(19.4) can be simplified to a convenient form if we assume the beam to possess the same flexural rigidity for all segments. If $(EI)_n = (EI)_{n+1}$ for all n, then, the above equation reduces to:

$$(M_{n-1})L_n + 2(L_n + L_{n+1})M_n + M_{n+1}L_{n+1} = -6\left(\frac{A_n\bar{x}_n}{L_n} + \frac{A_{n+1}\bar{x}_{n+1}}{L_{n+1}}\right) \tag{19.5}$$

[1]B.P.E. Clapeyron (1799–1864), a French engineer, developed three-moment equation in connection with the design of bridges.

Sometimes, the beam segments possess equal lengths also, that is, $L_n = L_{n+1}$ for all n, then, Eq. (19.5) reduces to more convenient form:

$$M_{n-1} + 4M_n + M_{n+1} = -\frac{6}{L^2}(A_n\bar{x}_n + A_{n+1}\bar{x}_{n+1}) \tag{19.6}$$

where it is assumed that all segments have equal length, L and possess identical flexural rigidity, EI.

In all Eqs. (19.4)–(19.6), we must remember that A_n and A_{n+1} represent the area under the bending moment diagrams due to the external loadings acting only on those members.

Having determined the bending moments, we can easily solve for support reactions. For example, reaction at B (point 'n') in Figure 19.5 can be found as

$$R_B = R_n\big|_{n\text{th member}} + R_n\big|_{(n+1)\text{th member}} \tag{19.7}$$

Let us now illustrate the use of the three-moment equations with the help of the following examples.

EXAMPLE 19.1

A continuous beam is shown with a single concentrated load in Figure 19.6. Solve the beam reactions and draw the shear force and bending moment diagrams to identify the maximum bending moment. Assume EI is constant.

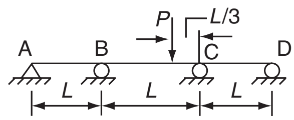


Figure 19.6 Example 19.1.

Solution

We draw the free-body diagram of the beam segments AB, BC and CD as shown in Figure 19.7:

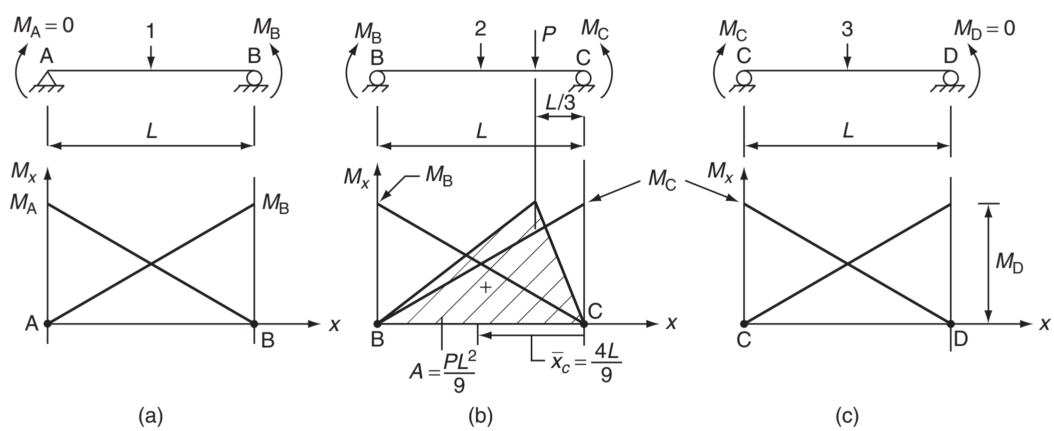


Figure 19.7 Moment diagrams of beam segments.

We note in the above segments, there are no external loadings on segments 2 and 3. In segment 2, there is a concentrated load P and the corresponding bending moment diagram is shown in Figure 19.7(b). The centroidal distance $\bar{x}_C$ of the bending moment diagram is marked also from point C. (Since EI is constant for all the beam segments, we just show the moment diagram instead of M/EI diagram as was shown in Figure 19.5.)

Since the segments possess equal EI and span length L, we can straightaway apply Eq. 19.6 for the *three-moment* equation successively for the segments 1, 2 and 3, keeping in mind that $M_A = 0$ and also $M_D = 0$ for segments 1 and 2. Clearly considering A, B and C as $n-1$, n and $n+1$, respectively, we get For segments 1 and 2:

$$M_A + 4M_B + M_C = -\frac{6}{L^2}\left(0 + \frac{-PL^2}{9} \times \frac{4L}{9}\right)$$

as $M_A = 0$, we get

$$\boxed{4M_B + M_C = -\frac{8PL}{27}} \qquad \textbf{(1)}$$

Again for segments 2 and 3:

$$M_B + 4M_C + M_D = -\frac{6}{L^2}\left(\frac{PL^2}{9} \times \frac{5L}{9} + 0\right)$$

as $M_D = 0$, we get

$$\boxed{M_B + 4M_C = -\frac{10PL}{27}} \qquad \textbf{(2)}$$

Solving Eqs. (1) and (2) simultaneously, we get

$$M_B = -\frac{22PL}{405} \quad \text{and} \quad M_C = -\frac{32PL}{405}$$

▶ **Note:** In Figure 19.7(b), the bending moment is

$$M = \frac{P}{L}\left(\frac{2L}{3}\right)\left(\frac{L}{3}\right) = \frac{2PL}{9}$$

Therefore, area is

$$A = \frac{1}{2} \times \frac{2PL}{9} \times L = \frac{PL^2}{9}$$

and centroidal distance shall be $4L/9$ from end C and $5L/9$ from end B.

Now, we are in a position to draw the bending moment diagram for the entire beam as shown in Figure 19.8. The figure also shows the variation of shear force along the beam length, which we can only find after determining the support reactions.

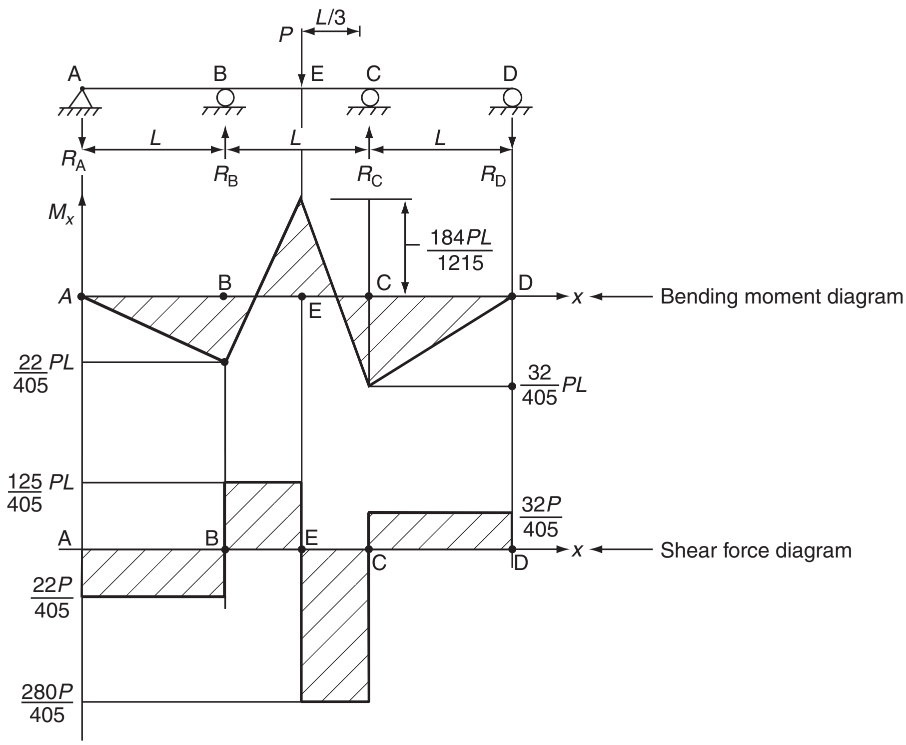


Figure 19.8 Shear force and bending moment diagrams.

We start with segment 2:

Clearly, R_B is the reaction force due to load P and moments M_B and M_C on segment 2. So,

$$R_B = \frac{P}{3} + \frac{M_C - M_B}{L} = \frac{P}{3} + \frac{1}{L}\left(-\frac{10PL}{40S}\right) = \frac{P}{3} - \frac{2P}{81}$$

or

$$R_B = \frac{25P}{81}(\uparrow) \quad \text{and} \quad R_C = P - R_B = \frac{56}{81}P(\uparrow) \tag{3}$$

Similarly, in segment 1:

$$R_B = \frac{M_B}{L} = \frac{22P}{405}(\uparrow) \quad \text{and} \quad R_A = \frac{22P}{405}(\downarrow) \tag{4}$$

Finally, in segment 3:

$$R_C = \frac{32P}{405}(\uparrow) \quad \text{and} \quad R_D = \frac{32P}{405}(\downarrow) \tag{5}$$

Adding the results of Eqs. (3)–(5), we get

$$R_A = \frac{22P}{405}(\downarrow),\; R_B = \frac{25P}{81} + \frac{22P}{405} = \frac{147P}{405}(\uparrow)$$

$$R_C = \frac{56P}{81} + \frac{32P}{405} = \frac{312}{405}P(\uparrow) \quad \text{and} \quad R_D = \frac{32P}{405}(\downarrow)$$

Note that $R_A + R_B + R_C + R_D = P$ as expected. Therefore, the required beam reactions are:

$$R_A = \frac{22P}{405}(\downarrow),\ R_B = \frac{147P}{405}(\uparrow),\ R_C = \frac{312P}{405}(\uparrow) \quad \text{and} \quad R_D = \frac{32P}{405}(\downarrow) \qquad \textbf{[Answer]}$$

From Figure 19.8, we conclude that

$$M_{\max} = \frac{184PL}{1215} = 0.1514PL$$

which occurs under the load P. **[Answer]**

EXAMPLE 19.2

Refer to Figure 19.9 of the continuous cantilever beam. Determine the beam reactions and the bending moments at points B and C. Also draw the shear force and bending moment diagrams to identify the maximum shear force and maximum bending moment.

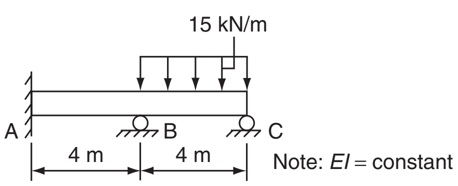


Figure 19.9 Example 19.2.

Solution

Let us consider the segments AB and BC as shown in Figure 19.10, where we consider the moment diagram only instead of *M/EI* diagram.

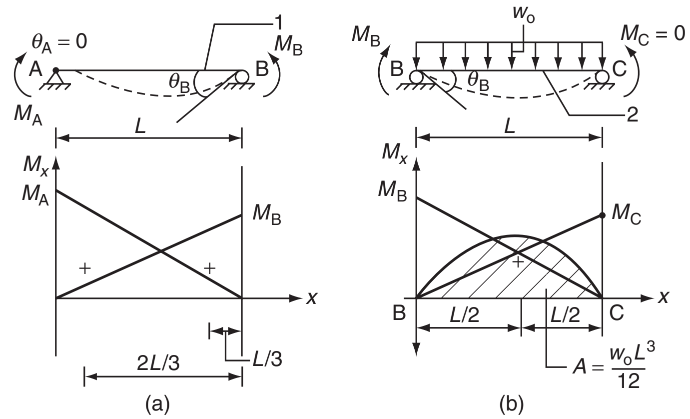


Figure 19.10 Moment diagrams.

We note for segment AB, $\theta_A = 0$ as A is the fixed-end of the beam. From second area-moment theorem, we can write:

$$t_{B/A} = 0$$

As tan $\theta_A = t_{B/A}/L$. Therefore,

$$\left(\frac{1}{2}\frac{M_B}{EI}L\right)\left(\frac{L}{3}\right)+\left(\frac{1}{2}\frac{M_A}{EI}\cdot L\right)\left(\frac{2L}{3}\right)=0$$

or

$$\frac{M_B}{6}+\frac{M_A}{3}=0 \Rightarrow 2M_A + M_B = 0 \tag{1}$$

Applying three-moment equation for segments AB and BC. As lengths are equal, we can apply Eq. (19.6) as

$$M_A + 4M_B + M_C = -\frac{6}{L^2}\left(0+\frac{w_o L^3}{12}\times\frac{L}{2}\right)$$

As end C is free, we can consider $M_C = 0$. So the above equation becomes

$$M_A + 4M_B = -\frac{w_o L^2}{4} \tag{2}$$

Solving Eqs. (1) and (2) simultaneously, we get

$$M_A = +\frac{w_o L^2}{28} \quad \text{and} \quad M_B = -\frac{w_o L^2}{14} \tag{3}$$

Having found the bending moments at A, B and C, we can now determine the support reactions as follows:
For segment AB:

$$R_A = \frac{M_B - M_A}{L} \quad \text{and} \quad R_B = \frac{M_A - M_B}{L} = -R_A$$

or

$$R_A = -\frac{3w_o L}{28} \quad \text{and} \quad R_B = \frac{3w_o L}{28} \tag{4}$$

Similarly, for segment BC, we find:

and

$$\begin{aligned} R_B &= \frac{w_o L}{2}+\frac{M_C - M_B}{L} = \frac{w_o L}{2}+\frac{w_o L}{14} = \frac{4w_o L}{7} \\ R_C &= \frac{w_o L}{2}+\frac{M_B - M_C}{L} = \frac{w_o L}{2}-\frac{w_o L}{14} = \frac{3w_o L}{7} \end{aligned} \tag{5}$$

Therefore, considering results from Eqs. (4) and (5) we get the support reactions as:

$$R_A = -\frac{3w_o L}{28}, \quad R_B = \frac{3w_o L}{28} + \frac{4w_o L}{7} = \frac{19w_o L}{28}$$

and $$R_C = \frac{3w_o L}{7}$$

Note that $R_A + R_B + R_C = w_o L$ (as expected). Now we can draw the shear force and bending moment diagrams as shown in Figure 19.11.

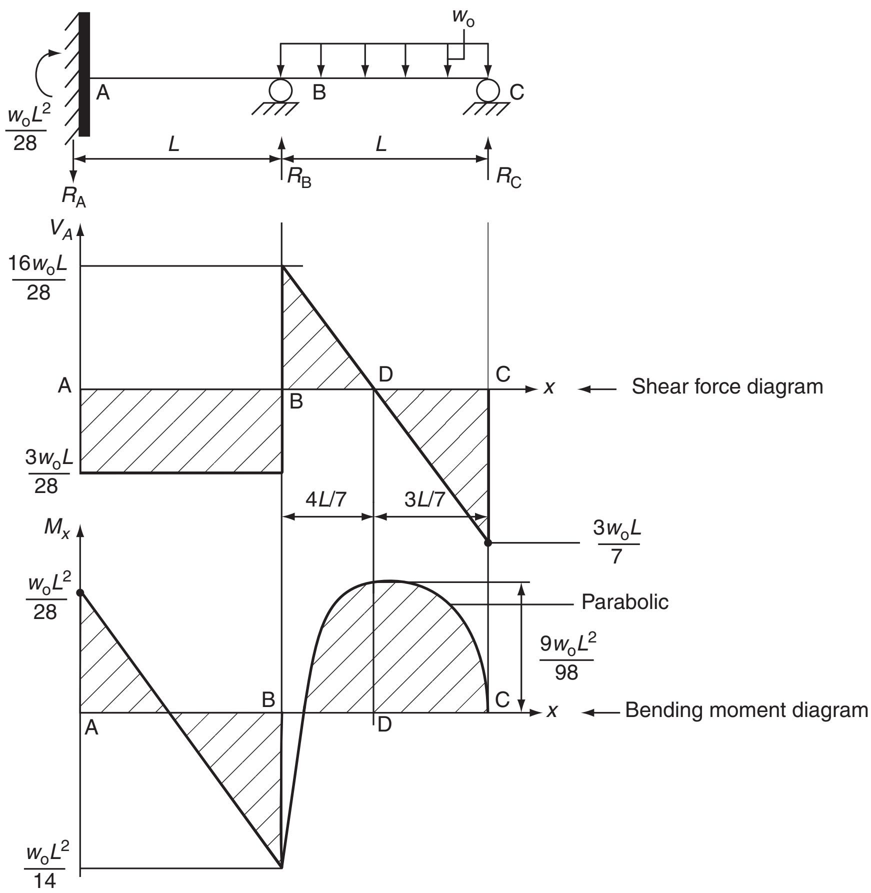


Figure 19.11 Shear force and bending moment diagrams.

From the diagram it is evident that

$$M_D = M_B + [\text{Area under shear force diagram between points B and D}]$$

Therefore,

$$M_D = -\frac{w_o L^2}{14} + \frac{1}{2}\left(\frac{4L}{7}\right)\left(\frac{16 w_o L}{28}\right)$$

$$= -\frac{w_o L^2}{14} + \frac{16 w_o L^2}{98}$$

$$= \frac{9 w_o L^2}{98}$$

so,
$$\boxed{M_{max} = \frac{9 w_o L^2}{98}} \qquad \textbf{(6)}$$

Now, putting numerical values $L = 4$ m and $w_o = 15$ kN/m, we get *support reactions* as*:*

$$R_A = -6.43 \text{ kN},\ R_B = +2.71 \text{ kN} \quad \text{and} \quad R_C = 25.71 \text{ kN}$$
$$M_A = 8.57 \text{ kN m},\ M_B = -17.14 \text{ kN m} \quad \text{and} \quad M_C = 0 \text{ kN m}$$

The maximum shear force is 34.29 kN and maximum bending moment is 22.04 kN. **[Answer]**

EXAMPLE 19.3

Construct the shear force and bending moment diagrams for the continuous beam shown in Figure 19.12. Assume EI = constant.

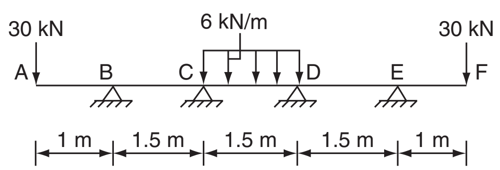


Figure 19.12 Example 19.3.

Solution

Five segments of the given beam are considered as shown in Figure 19.13:

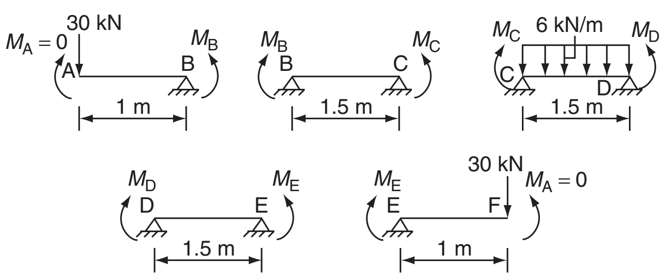


Figure 19.13 Beam segments.

Clearly, we can consider moments at B and E as 30 kN m as shown in Figure 19.14 (since there is no external loading on BC and DE, we have not drawn their moment diagrams).

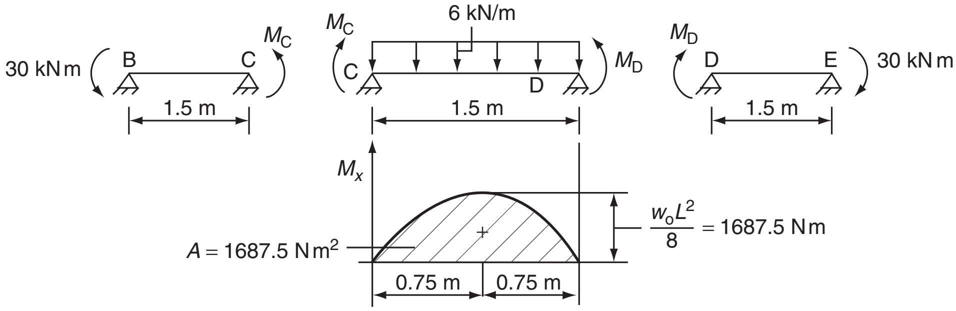


Figure 19.14 Moment of beam segments.

Applying three-moment equation between the segments BC and CD:

$$M_B + 4M_C + M_D = -\frac{6}{1.5^2}(0 + 1687.5 \times 0.75)$$

or $$M_B + 4M_C + M_D = -3375 \text{ N m}$$

Putting $M_B = -30000$ N m, the above equation becomes

$$4M_C + M_D = 26625 \text{ N m} \quad \textbf{(1)}$$

Again, looking at the symmetry of the problem, $M_C = M_D$. Therefore,

$$M_C = M_D = 5325 \text{ N m} = 5.325 \text{ kN m}$$

Also, $M_B = M_E = -30$ kN m and $M_A = M_F = 0$. Now, considering beam segment AB, we get

$$R_B = +30 \text{ kN}$$

again taking beam segment BC, we get

$$R_B = \frac{M_C - M_B}{L} = \frac{5.325 - (-30)}{1.5} = 23.55 \text{ kN}$$

Therefore, by considering $R_B = R_B\big|_{AB} + R_B\big|_{BC}$, we get

$$R_B = 53.55 \text{ kN}$$

Similarly, considering member CD, we get

$$R_C = 4.5 + \frac{M_D - M_C}{L} = 4.5 + \frac{5.325 - 5.325}{1.5} = 4.5\ \text{kN}$$

or $$R_C = R_C\big|_{BC} + R_C\big|_{CD} = (-23.55 + 4.5)\ \text{kN} = -19.05\ \text{kN}$$

From symmetry, $R_D = R_C = -19.05$ kN and $R_E = R_B = 23.55$ kN. Let us now draw the shear force and bending moment diagrams as shown in Figure 19.15:

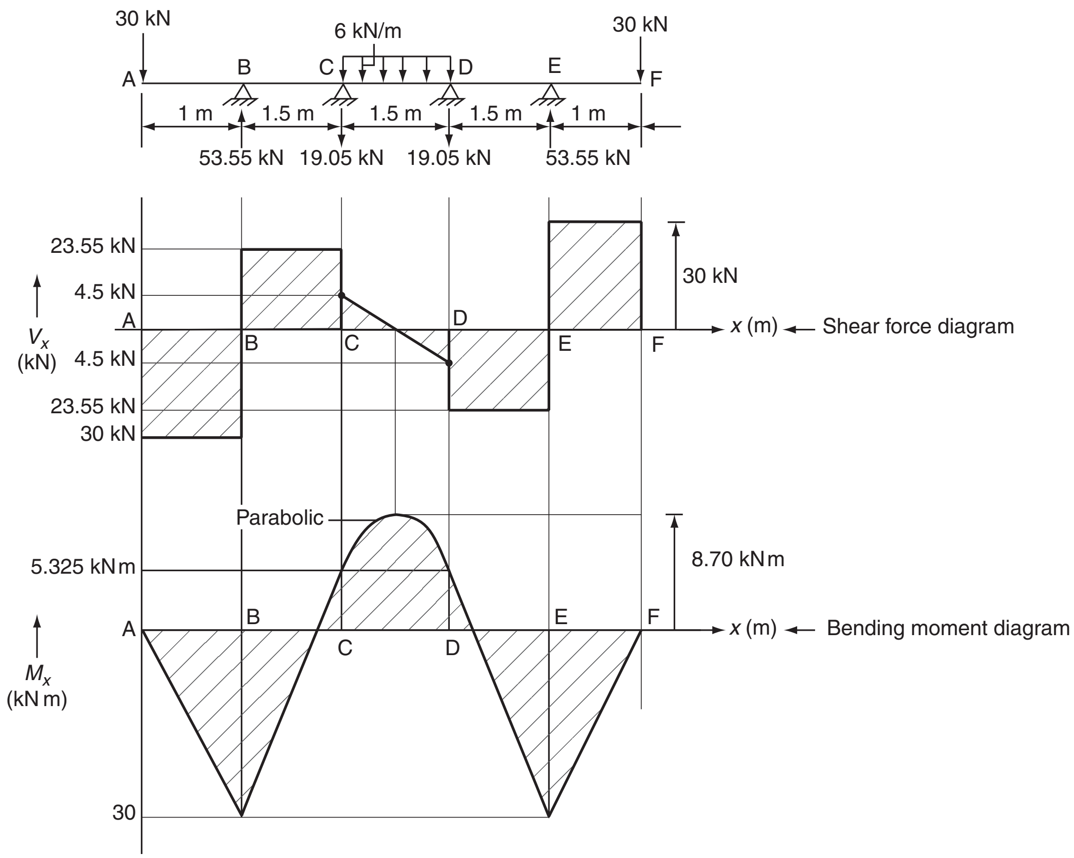


Figure 19.15 Shear force and bending moment diagrams.

[Answer]

EXAMPLE 19.4

Refer to Figure 19.16 of the continuous beam. Assuming flexural rigidity to be constant, determine the support reactions and draw the shear force and bending moment diagrams.

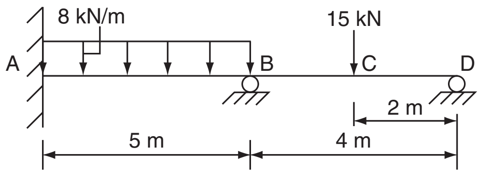


Figure 19.16 Example 19.4.

Solution

We show the two segments AB and BD noting their different spans in Figure 19.17:

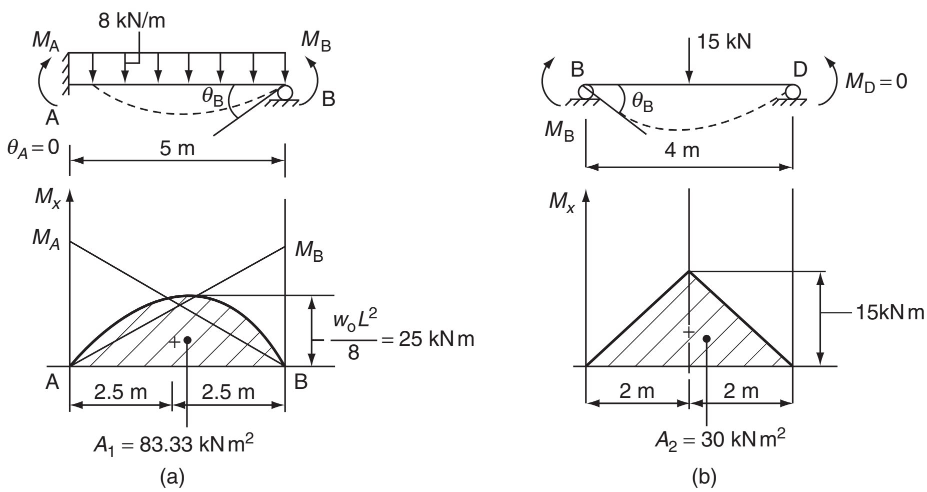


Figure 19.17 Moment diagrams of beam segments.

For *segment AB*:

Noting $\theta_A = 0$ for beam segment AB and applying second area-moment theorem, we get:

$$t_{B/A} = 0 \Rightarrow \frac{250}{3} \times \frac{5}{2} + \left(\frac{1}{2} M_B \times 5\right)\left(\frac{5.0}{3}\right) + \frac{1}{2}(M_A \times 5)\left(\frac{2 \times 5}{3}\right) = 0$$

$$\Rightarrow \frac{25M_A}{3} + \frac{25M_B}{6} = -\frac{(250)(5)}{6}$$

or $$2M_A + M_B = -50 \text{ kN m} \qquad \textbf{(1)}$$

Now, applying three-moment theorem for segments AB and BD above by applying Eq. 19.3(b) as $L_{AB} \neq L_{BD}$, we get

$$5M_A + 2(5+4)M_B + 4M_D = -6\left(\frac{250}{3} \times \frac{5}{2} \times \frac{1}{5} + 30 \times 2 \times \frac{1}{4}\right)$$

as $M_D = 0$, the above equation becomes

$$5M_A + 18M_B = -340 \text{ kN m} \qquad \textbf{(2)}$$

Solving Eqs. (1) and (2), we get

$$M_A = -18.06 \text{ kN m} \quad \text{and} \quad M_B = -13.87 \text{ kN m}$$

Having determined bending moments at A and B, we can proceed to find the various support reactions. To this end, we take up the segments AB and BD once again. Thus, for the *segment AB*:

$$R_A = \frac{(w_o)(L)}{2} + \frac{M_B - M_A}{L} = \frac{(8)(5)}{2} + \frac{-13.87 + 18.06}{5}$$
$$= 20.838 \text{ kN}(\uparrow)$$

Similarly,

$$R_B = \frac{w_o L_{AB}}{2} + \frac{M_A - M_B}{L_{AB}} = \frac{(8)(5)}{2} + \frac{-18.06 + 13.87}{5}$$
$$= 19.162 \text{ kN}(\uparrow)$$

Again, for *segment BD*:

$$R_B = \frac{P}{2} + \frac{M_D - M_B}{L_{BD}} = \frac{15}{2} + \frac{0 - (-13.87)}{4}$$
$$= 10.9675 \text{ kN}(\uparrow)$$

and $$R_D = \frac{P}{2} + \frac{M_B - M_D}{L_{BD}} = \frac{15}{2} + \frac{-13.87 - 0}{4}$$

$$= 4.0325 \text{ kN}(\uparrow)$$

Thus, adding results of the segments, we get

$$R_A = 20.838 \text{ kN}(\uparrow),\ R_B = 30.1295 \text{ kN}(\uparrow) \text{ and } R_D = 4.0325 \text{ kN}(\uparrow)$$

also $$M_A = -18.06 \text{ kN m}$$

[Answer]

Let us now proceed to draw the shear force and bending moment diagrams of the beam, incorporating reaction forces in their true senses in Figure 19.18.

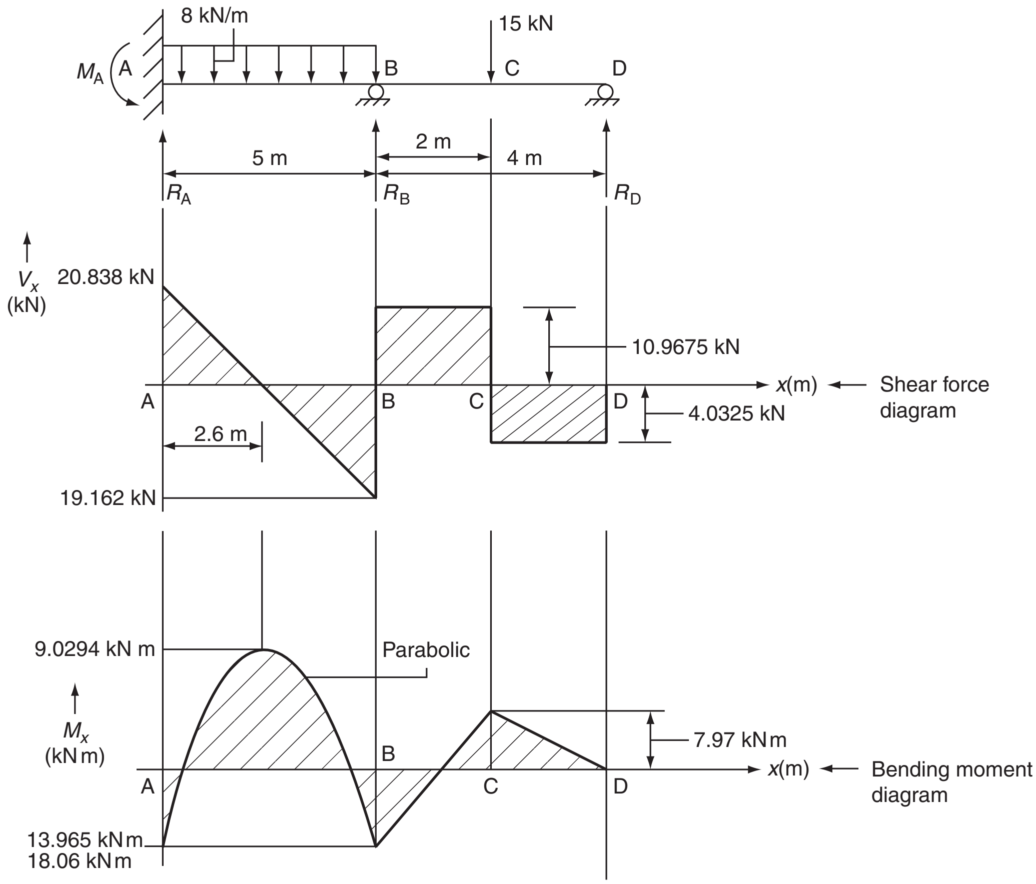


Figure 19.18 Shear force and bending moment diagrams.

[Answer]

EXAMPLE 19.5

Solve completely the continuous beam shown in Figure 19.19 and draw the shear force and bending moment diagrams.

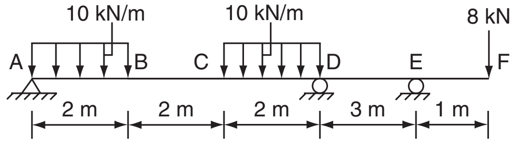


Figure 19.19 Example 19.5.

Solution

Let us first consider the equivalent loading for the given beam as shown in Figure 19.20:

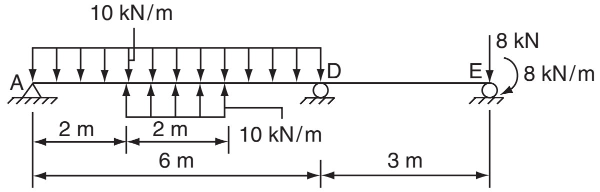


Figure 19.20 Equivalent loading.

Let us now consider the two segments as shown in Figure 19.21 along with the moment diagrams:

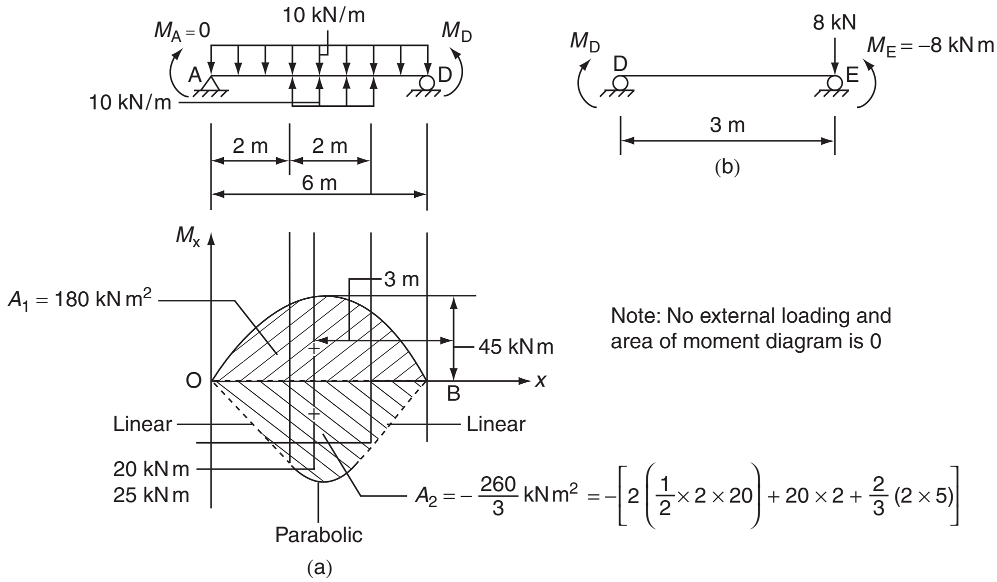


Figure 19.21 Moment diagrams of beam segments.

Now, applying three-moment theorem for the segments considering $M_A = 0$ and $M_E = -8$ kN m, we get using the Eq. (19.5):

$$(0)(6) + 2(6+3)M_D + 3(-8) = -6\left[\frac{\{180-(260/3)\}(3)}{6} + 0\right] = -280$$

or

$$M_D = -14.22 \text{ kN m}$$

Hence, bending moments at the relevant points are

$$M_A = 0 \text{ kN m}, \quad M_D = -14.22 \text{ kN m} \quad \text{and} \quad M_E = -8 \text{ kN m}$$

Correspondingly, the support reactions at various points are calculated by considering the two segments separately. Accordingly, if we take the *segment AD*:

$$R_{\mathrm{A}} = \frac{(10)(6)}{2} + \frac{(-10)(2)}{2} + \frac{-14.22 - 0}{6}\,\mathrm{kN} = 17.63\ \mathrm{kN}(\uparrow)$$

and
$$R_{\mathrm{D}} = \frac{(10)(6)}{2} + \frac{(-10)(2)}{6} + \frac{0-(-14.22)}{6}\,\mathrm{kN} = 22.37\ \mathrm{kN}(\uparrow)$$

Similarly, considering *segment DE*:

$$R_{\mathrm{D}} = 0 + \frac{M_{\mathrm{E}} - M_{\mathrm{D}}}{L_{\mathrm{DE}}} = \frac{-8-(-14.22)}{3}\,\mathrm{kN} = 2.07\ \mathrm{kN}(\uparrow)$$

and
$$R_{\mathrm{E}} = 8 + \frac{-14.22-(-8)}{3} = 5.93\ \mathrm{kN}(\uparrow)$$

Therefore, the support reactions are: $= 17.63\ \mathrm{kN}(\uparrow)$, $R_{\mathrm{D}} = 24.44\mathrm{kN}(\uparrow)$ and $R_{\mathrm{E}} = 5.93\mathrm{kN}(\uparrow)$.

[Answer]

[Note that $R_{\mathrm{A}} + R_{\mathrm{D}} + R_{\mathrm{E}} = (17.63 + 24.44 + 5.93)\ \mathrm{kN} = 48\ \mathrm{kN}$ as expected.]

Now, we are in a position to draw the shear force and bending moment diagrams of the original beam as shown in Figure 19.22:

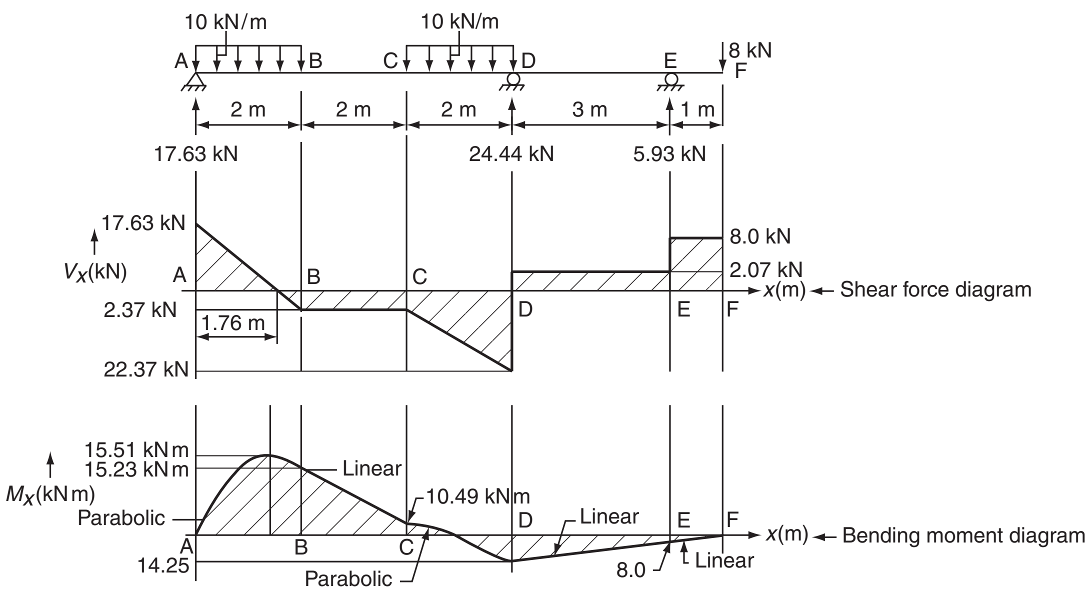


Figure 19.22 Shear force and bending moment diagrams.

[Answer]

Summary

In this chapter, we initially discussed about statically indeterminate beams. Although we have come across such beams in our earlier chapters also, a methodical analysis was lacking to determine the support reactions in terms of shear force and bending moments at various locations on the beam. In that regard, we have carefully introduced three-moment theorem to analyse such cases in a more elegant manner. Various forms of the equation are presented to cope up with different conditions of physical arrangements of beams.

Key Terms

Statically determinate beam
Statically indeterminate beam
Degree of indeterminacy
Continuous beam
Principle of superposition
Linear structure
Second area-moment theorem
Clapeyron's three-moment equation
Shear force diagram
Bending moment diagram

Review Questions

1. What do you mean by statically indeterminate beam?
2. Explain the degree of indeterminacy of statically indeterminate beam.
3. What is a continuous beam?
4. Explain the second theorem of area moment.
5. What do you mean by three-moment equation?
6. Derive Clapeyron's three-moment equation.

Numerical Problems

1. A uniform span continuous beam is shown in Figure 19.23 with equal overhangs. Find $a{:}L$ such that $M_B = M_C = M_D$.

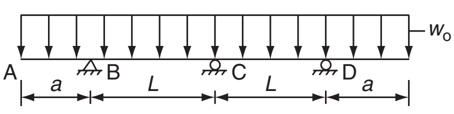


Figure 19.23 Problem 1.

2. For the above problem, find $a{:}L$ such that, $R_B = R_C = R_D$.

3. For the cantilever beam shown in Figure 19.24, determine the support reactions at the fixed-end of the beam.

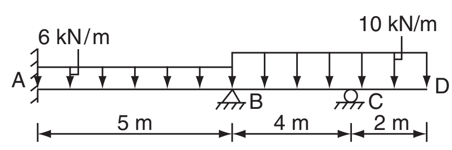


Figure 19.24 Problem 3.

4. Calculate the support reactions at points A and D of the beam shown in Figure 19.25.

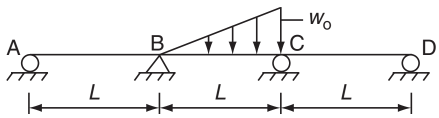


Figure 19.25 Problem 4.

[*Hint*:

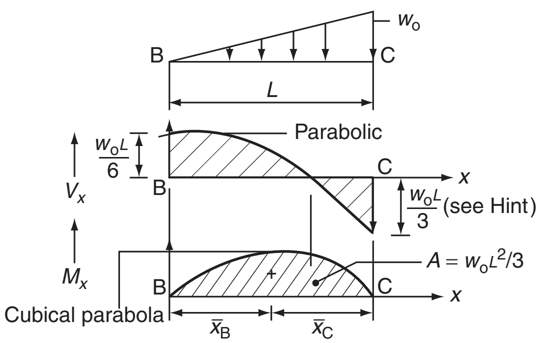


Figure 19.26 Hint for Problem 4.

If we take segment BC, the shear force and bending moment diagram shown in Figure 19.26 results due to the given external loading.

You can easily show the equation of parabolic shear force variation, as

$$V_x = \frac{w_o L}{6} - \left(\frac{w_o}{2L}\right)x^2 = \frac{dM_x}{dx}$$

Therefore,

$$M_x = \left(\frac{w_o L}{6}\right)x - \left(\frac{w_o}{6L}\right)x^3$$

The area A in the figure is

$$A = \int_0^L M_x \, dx = \frac{w_o L^3}{24}$$

Also,

$$\bar{x}_B = \frac{\int_0^L x M_x dx}{A} = \frac{8L}{15} \quad \text{and} \quad \bar{x}_C = \frac{7L}{15}.$$

Taking segments AB, BC and CD and applying the three-moment equation, Eq. (19.6) we get

$$M_A = 0,\ M_B = -\frac{w_o L^2}{45},$$

$$M_C = -\frac{w_o L^2}{36} \quad \text{and} \quad M_D = 0$$

Thus clearly, $R_A = -w_o L/45$ and $R_D = -w_o L/36$ from the free-body diagrams of segment AB and CD.]

5. A beam of length L carries a uniformly distributed load w_o/unit length and rests on three supports, two at the ends and one in the middle. Find how much the middle support is lower than the end ones in order that the pressures on the three supports are equal.

6. Calculate the moments acting at A and B for the following beam shown in Figure 19.27:

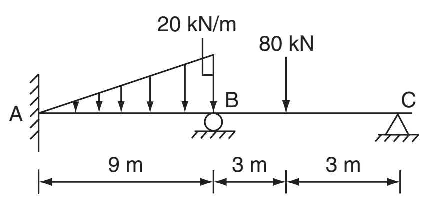


Figure 19.27 Problem 6.

Answers

Numerical Problems

1. $1/\sqrt{6}$
2. 0.44
3. $R_A = 15.45$ kN; $M_A = -16.49$ kN m
4. $R_A = -w_o L/45$ and $R_D = -w_o L/36$
5. Middle support $= \dfrac{3}{384}\dfrac{w_o L^4}{EI}$
6. $M_A = 51.9$ kN m; $M_B = 0$ kN m