MPL's Algebraic_expand Operator

In an algebraic sense, MPL's *Algebraic_expand* operator applies the two distributive transformations $(a+b)\,c = a\,c + b\,c$ and $a\,(b+c) = a\,b + a\,c$ in a left to right fashion to products and powers that contain sums. With these transformations, the operator obtains manipulations such as:

$$\begin{aligned}
(x+2)\,(x+3)\,(x+4) \quad &\to \quad x^3 + 9\,x^2 + 26\,x + 24, && (1)\\
(x+y+z)^3 \quad &\to \quad x^3 + y^3 + z^3 + 3\,x^2\,y + 3\,x^2\,z + 3\,y^2\,x \\
&\qquad + 3\,y^2\,z + \ 3\,z^2\,x + 3\,z^2\,y + 6\,x\,y\,z, && (2)\\
(x+1)^2 + (y+1)^2 \quad &\to \quad x^2 + 2\,x + y^2 + 2\,y + 2, && (3)\\
((x+2)^2 + 3)^2 \quad &\to \quad x^4 + 8\,x^3 + 30\,x^2 + 56\,x + 49. && (4)
\end{aligned}$$

The last two examples show that *Algebraic_expand* is recursive.

There are, however, other instances where it is less certain what the operator should do. For example, should *Algebraic_expand* perform the following manipulations?

$$\begin{aligned}
\frac{a}{(x+1)\,(x+2)} \quad &\to \quad \frac{a}{x^2 + 3\,x + 2}, && (5)\\
(x+y)^{3/2} \quad &\to \quad x\,(x+y)^{1/2} + y\,(x+y)^{1/2}. && (6)
\end{aligned}$$

The first example differs from those above because a denominator contains a product of sums, while the second example involves non-integer exponents.

The next definition gives the form of the output of our *Algebraic_expand* operator.

Definition 1 *An algebraic expression u is in* **expanded form** *if the set Variables(u) does not contain a sum.*

According to this definition, the expressions on the left in (1)-(4) are in unexpanded form, while those on the right are in expanded form. For example,

$$\mathit{Variables}((x+2)\,(x+3)\,(x+4)) \to \{x+2, x+3, x+4\},$$

while for the expanded form of this expression,

$$\mathit{Variables}(x^3 + 9\,x^2 + 26\,x + 24) \to \{x\}.$$

On the other hand, the expressions on the left in (5) and (6) are already in expanded form, and so our *Algebraic_expand* operator does not obtain the manipulations shown for these expressions.

Definition 1 only makes sense if it is understood in the context of automatic simplification. Without this context, some expressions that are obviously not in expanded form satisfy the definition. For example, $u = \left((x+1)^2\right)^2$ is certainly not in expanded form, and since automatic simplification obtains the transformation

$$\left((x+1)^2\right)^2 \rightarrow (x+1)^4, \tag{7}$$

we have *Variables* $(u) = \{x+1\}$. On the other hand, without the transformation (7), *Variables*$(u) = \{(x+1)^2\}$, which does not contain a sum.

An MPL algorithm for *Algebraic_expand* is given in the companion book, *Computer Algebra and Symbolic Computation, Elementary Algorithms* (Cohen [24], pages 250-257).

Most computer algebra systems have an operator that is similar to *Algebraic_expand*:

MPL	Maple	Mathematica	MuPAD
Algebraic_expand(u)	`expand(u)`	`Expand[u]`	`expand(u)`

(Implementation: Maple (mws), Mathematica (nb), MuPAD (mnb).)

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