General Polynomial Expressions

Single Variable Polynomials

We begin by considering polynomials in a single variable with rational number coefficients.

Definition 1 *A* **polynomial** *u in a single variable x is an expression of the form:*

$$u = u_n x^n + u_{n-1} x^{n-1} + \ldots + u_1 x + u_0, \tag{1}$$

where the **coefficients** *u_j are rational numbers, and n is a non-negative integer. If $u_n \neq 0$, then u_n is called the* **leading coefficient** *of u and n is its* **degree**. *The expression $u = 0$ is called the* **zero polynomial***; it has leading coefficient 0 and, according to mathematical convention has degree $-\infty$. The leading coefficient is represented by $\mathrm{lc}(u, x)$ and the degree by $\deg(u, x)$. When the variable x is evident from context, we use the simpler notations $\mathrm{lc}(u)$ and $\deg(u)$.*

Observe that we have distinguished the zero polynomial from other constant polynomials because it has no non-zero coefficients, and so the general definitions for leading coefficient and degree do not apply[1].

Example 1

$$u = 3x^6 + 2\,x^4 - 5/2, \quad \deg(u) = 6, \quad \mathrm{lc}(u) = 3,$$

$$u = x^2 - x + 2, \quad \deg(u) = 2, \quad \mathrm{lc}(u) = 1, \tag{2}$$

$$u = 2x^3, \quad \deg(u) = 3, \quad \mathrm{lc}(u) = 2, \tag{3}$$

$$u = 3, \quad \deg(u) = 0, \quad \mathrm{lc}(u) = 3. \tag{4}$$

Multivariate Polynomials

Polynomials that contain more than one variable are called *multivariate polynomials*.

[1] For the polynomial $u = 0$, both Maple's `degree` operator and Mathematica's `Exponent` operator return a degree of $-\infty$. On the other hand, MuPAD's degree operator returns a degree of 0.

Definition 2 *A* **multivariate polynomial** *u in the symbols $\{x_1, x_2, \ldots, x_m\}$ is a finite sum with (one or more) monomial terms of the form*

$$c\, x_1^{n_1}\, x_2^{n_2} \cdots x_m^{n_m},$$

where the coefficient c is a rational number and the exponents n_j are non-negative integers.

Example 2 The following are multivariate polynomials:

$$p + 1/2\, \rho\, v^2 + \rho\, g\, y, \quad a\, x^2 + 2\, b\, x + 3\, c, \quad x^2 - y^2, \quad m\, c^2, \quad 3\, x^2 + 4.$$

General Polynomial Expressions

There are many expressions that are polynomials in a computational context that are not included in previous definitions for polynomials. For example, it is reasonable to consider the expression $u = \dfrac{a}{(a+1)} x^2 + b\, x + \dfrac{1}{a}$ as a polynomial in x, even though it does not satisfy the above definitions. Indeed, a CAS views this expression as a polynomial when it solves the quadratic equation $u = 0$ for x. In addition, it is reasonable to view the expressions $\sin^3(x) + 2\, \sin^2(x) + 3$ and $(x+1)^3 + 2\,(x+1)^2 + 3$ as polynomials in terms of a complete sub-expression ($\sin(x)$ or $(x+1)$). On the other hand, the expression $(3 \sin(x)) x^2 + (2 \ln(x)) x + 4$ is not a polynomial in x because the coefficients of the powers of x also depend on x.

The next definition includes the more general polynomial expressions given above.

Definition 3 *Let $c_1, c_2, \ldots, c_r$ be algebraic expressions, and let $x_1, x_2, \ldots, x_m$ be algebraic expressions that are not integers or fractions. A* **general monomial expression** *(GME) in $\{x_1, x_2, \ldots, x_m\}$ is an expression of the form*

$$c_1\, c_2 \cdots c_r\, x_1^{n_1}\, x_2^{n_2} \cdots x_m^{n_m}, \tag{5}$$

where the exponents n_j are non-negative integers and each c_i satisfies the independence property

$$\mathit{Free\_of}(c_i, x_j) \to \textbf{true}, \quad \text{for } j = 1, 2, \ldots, m. \tag{6}$$

The expressions x_j are called **generalized variables** *because they mimic the role of variables, and the expressions c_i are called* **generalized coefficients** *because they mimic the role of coefficients. The expression*

$$x_1^{n_1} \cdots x_m^{n_m}$$

is called the **variable part** *of the monomial, and if there are no generalized variables in the monomial, the variable part is 1. The expression $c_1 \cdots c_r$ is called the* **coefficient part** *of the monomial, and if there are no generalized coefficients in the monomial, the coefficient part is 1. An expression u is a* **general polynomial expression** *(GPE) if it is either a GME or a sum of GMEs in $\{x_1, x_2, \ldots, x_m\}$.*

Example 3 The following are general polynomial expressions:

$$x^2 - x + 1, \quad (x_1 = x),$$
$$x^2\, y - x\, y^2 + 2, \quad (x_1 = x,\, x_2 = y),$$
$$\frac{a}{(a+1)} x^2 + b\, x + \frac{1}{a}, \quad (x_1 = x), \tag{7}$$
$$\sin^3(x) + 2\, \sin^2(x) + 3, \quad (x_1 = \sin(x)), \tag{8}$$
$$(x+1)^3 + 2\,(x+1)^2 + 3, \quad (x_1 = x+1), \tag{9}$$
$$\sqrt{2}\, x^2 + \sqrt{3}\, x + \sqrt{5}, \quad (x_1 = x). \tag{10}$$

The definition is quite general. It includes the single variable polynomials (Definition 1), multivariate polynomials (Definition 2) and allows more general expressions ((7), (8), (9), and (10)). Notice that (10) is a GPE, but not a single variable polynomial in the sense of Definition 1 because the coefficients are not rational numbers. On the other hand, the expression $(\sin(x))\, x^2 + (\ln(x))\, x + 4$ is not a GPE in x alone because the coefficients $\sin(x)$ and $\ln(x)$ do not satisfy the independence property (6).

The definition is also quite flexible because it allows for a choice of which parts of an expression act as variables and which parts act as coefficients. For example, the expression $2\, a\, x^2 + 3\, b\, x + 4\, c$ can be viewed as a polynomial in $\{a, b, c, x\}$ with integer coefficients or as a polynomial in x with coefficients $2\, a$, $3\, b$ and $4\, c$. In fact, it is possible to view the expression as a polynomial in another variable (say z) with the entire expression as the coefficient part of z^0. In addition, since a sum can be a generalized variable, we can even designate the entire expression as a generalized variable and view it as a polynomial in terms of itself.

The following definitions for a GME and GPE are more suitable for computational purposes.

Definition 4 **(Computational Definition)** *A* **general monomial expression** *(GME) in a set of generalized variables*

$$S = \{x_1, x_2, \ldots, x_m\}$$

is an algebraic expression that satisfies one of the following rules.

GME-1. *Free_of*$(u, x_j) \rightarrow$ **true**, *for* $j = 1, \ldots, m$.

GME-2. $u \in S$.

GME-3. $u = x^n$, *where* $x \in S$ *and* $n > 1$ *is an integer.*

GME-4. u *is a product, and each operand of* u *is a GME in* S.

A **general polynomial expression** *(GPE) in a set* S *of expressions is an algebraic expression* u *that satisfies one of the following rules.*

GPE-1. *u is a GME in S.*

GPE-2. *u is a sum and each operand of u is a GME in S.*

Primitive Operations for General Polynomial Expressions

The operators described in the following definitions obtain the polynomial structure of an expression.

The *Monomial_gpe* and *Polynomial_gpe* Operators

Definition 5 *Let u be an algebraic expression, and let v be either a generalized variable x or a set S of generalized variables. The operator*

$$Monomial\_gpe(u, v)$$

returns **true** *whenever u is a GME in $\{x\}$ or in S, and otherwise returns* **false**. *The operator*

$$Polynomial\_gpe(u, v) \tag{11}$$

returns **true** *whenever u is a GPE in $\{x\}$ or in S, and otherwise returns* **false**.

Example 4

$$\begin{array}{rcl}
Monomial_gpe(a\,x^2\,y^2,\ \{x,\,y\}) & \rightarrow & \textbf{true}, \\
Monomial_gpe(x^2 + y^2,\ \{x,\,y\}) & \rightarrow & \textbf{false}, \\
Polynomial_gpe(x^2 + y^2,\ \{x,\,y\}) & \rightarrow & \textbf{true}, \\
Polynomial_gpe(\sin^2(x) + 2\,\sin(x) + 3,\ \sin(x)) & \rightarrow & \textbf{true}, \\
Polynomial_gpe(x/y + 2\,y,\ \{x,\,y\}) & \rightarrow & \textbf{false}, \\
Polynomial_gpe((x+1)\,(x+3),\ x) & \rightarrow & \textbf{false}.
\end{array}$$

The *Variables* Operator

The polynomial structure of a GPE depends on which expressions are chosen for the generalized variables. The operator in the next definition defines a natural set of generalized variables for an expression.

Definition 6 *Let u be an algebraic expression. The operator*

$$Variables(u)$$

is defined by the following transformation rules.

VAR-1. *If u is an integer or fraction, then*

$$Variables(u) \rightarrow \emptyset.$$

VAR-2. *Suppose u is a power. If the exponent of u is an integer that is greater than 1, then*

$$Variables(u) \to \{\,Operand(u,1)\} \qquad \text{(the base of } u\text{)},$$

otherwise

$$Variables(u) \to \{u\}.$$

VAR-3. *Suppose u is a sum. Then $Variables(u)$ is the union of the generalized variables of each operand of u obtained using rules VAR-1, VAR-2, VAR-4, or VAR-5.*

VAR-4. *Suppose u is a product. Then $Variables(u)$ contains the union of the generalized variables of each operand of u determined by rules VAR-1, VAR-2, or VAR-5, as well as any operand that is a sum.*

Observe that for a product we include an operand that is a sum in the variable set (see (12) below) even though a sum by itself is not in the variable set (VAR-3).

VAR-5. *If u is not covered by the above rules, then*

$$Variables(u) \to \{u\}.$$

The last rule covers symbols, function forms, and factorials.

Example 5 For a multivariate polynomial, the operator returns the set of variables in the expression:

$$Variables(x^3 + 3\,x^2\,y + 3\,x\,y^2 + y^3) \to \{x, y\}.$$

Other examples include

$$Variables(3\,x\,(x+1)\,y^2 z^n) \to \{x,\ x+1,\ y,\ z^n\}, \tag{12}$$

$$Variables(a\sin^2(x) + 2\,b\sin(x) + 3\,c) \to \{a,\ b,\ c,\ \sin(x)\},$$

$$Variables(1/2) \to \emptyset,$$

$$Variables(\sqrt{2}\,x^2 + \sqrt{3}\,x + \sqrt{5}) \to \{x, \sqrt{2}, \sqrt{3}, \sqrt{5}\}.$$

The last example shows that the *Variables* operator also selects expressions that do not vary in the mathematical sense, but still act as natural place holders in the expression. In fact, any *algebraic expression* u is always a GPE in terms of $Variables(u)$, and when it is viewed in this way, the coefficient part in each monomial is an integer or fraction.

The *Degree_gpe* Operator

In the next definition we generalize the degree concept to generalized polynomial expressions.

Definition 7 *Let $S = \{x_1, \ldots, x_m\}$ be a set of generalized variables. Let*

$$u = c_1 \cdots c_r \cdot x_1^{n_1} \cdots x_m^{n_m}$$

be a monomial with non-zero coefficient part. The **degree** *of u with respect to the set S is the sum of the exponents of the generalized variables:*

$$\deg(u, S) = n_1 + n_2 + \cdots + n_m.$$

By mathematical convention, the degree of the 0 monomial is defined to be $-\infty$.

If u is a GPE that is a sum of monomials, then $\deg(u, S)$ is the maximum of the degrees of the monomials. If S contains a single generalized variable x, we use the simpler notation $\deg(u, x)$, and if the generalized variables are understood from context, we use $\deg(u)$.

Example 6

$$\deg(3\, w\, x^2\, y^3\, z^4,\ \{x,\ z\}) = 6,$$

$$\deg(a\, x^2 + b\, x + c,\ x) = 2,$$

$$\deg(a \sin^2(x) + b \sin(x) + c,\ \ \sin(x)) = 2,$$

$$\deg(2\, x^2\, y\, z^3 + w\, x\, z^6,\ \{x\, , z\}) = 7.$$

Definition 8 *Let u be an algebraic expression, and let v be a generalized variable x or a set S of generalized variables. The degree operator has the form:*

$$\textit{Degree\_gpe}(u, v).$$

When u is a GPE in v, the operator returns $\deg(u, v)$. If u is not a GPE in v, the operator returns the global symbol **Undefined***.*

The *Coefficient_gpe* Operator

Definition 9 *Let u be an algebraic expression. If u is a GPE in a generalized variable x and $j \geq 0$ is an integer, then the operator*

$$\textit{Coefficient\_gpe}(u, x, j)$$

returns the sum of the coefficient parts of all monomials of u with variable part x^j. If there is no monomial with variable part x^j, the operator returns 0. If u is not a polynomial in x, the operator returns the global symbol **Undefined***.*

Example 7

$$
\begin{aligned}
\mathit{Coefficient\_gpe}(a\,x^2 + b\,x + c,\; x,\; 2) &\;\rightarrow\; a, \\
\mathit{Coefficient\_gpe}(3\,x\,y^2 + 5\,x^2 y + 7\,x + 9,\; x,\; 1) &\;\rightarrow\; 3\,y^2 + 7, \\
\mathit{Coefficient\_gpe}(3\,x\,y^2 + 5\,x^2 y + 7\,x + 9,\; x,\; 3) &\;\rightarrow\; 0, \\
\mathit{Coefficient\_gpe}((3\,\sin(x))\,x^2 + (2\,\ln(x))\,x + 4,\; x,\; 2) &\;\rightarrow\; \mathbf{Undefined}.
\end{aligned}
$$

The *Leading_coefficient_gpe* Operator

Definition 10 *Let u be an algebraic expression. If u is a* GPE *in x, then the leading coefficient of u with respect to x is defined as the sum of the coefficient parts of all monomials with variable part $x^{\deg(u,x)}$. The leading coefficient is represented by* $\mathrm{lc}(u,x)$*, and when x is understood from context, by the simpler notation* $\mathrm{lc}(u)$.

For example, $\mathrm{lc}(3\,x\,y^2 + 5\,x^2 y + 7\,x^2\,y^3 + 9, x) = 5\,y + 7\,y^3$.

Definition 11 *Let u be a* GPE *in x. The operator*

$$\mathit{Leading\_coefficient\_gpe}(u, x)$$

returns $\mathrm{lc}(u,x)$. *If u is not a* GPE *in x, the operator returns* **Undefined**.

A general approach for developing algorithms for the polynomial operators in terms of the primitive structural operators is described in the companion book, *Computer Algebra and Symbolic Computation, Elementary Algorithms* (Cohen [24], Chapter 6).

The operators in computer algebra systems that correspond most closely to MPL's polynomial operators are given in Figure 1.

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MPL	Maple	Mathematica	MuPAD
Polynomial_gpe(u, x)	`type(u,polynom(anything,x))`	`PolynomialQ[u,x]`	`testtype(u, Type::PolyExpr(x))`
Degree_gpe(u, x)	`degree(u,x)`	`Exponent[u,x]`	`degree(u,x)`
Coefficient_gpe(u, x, n)	`coeff(u,x,n)`	`Coefficient[u,x,n]`	`coeff(u,x,n)`
Leading_coefficient_gpe(u, x, n)	`lcoeff(u,x)`	`Coefficient[u,x,Exponent[u,x]]`	`lcoeff(u,x)`
Variables(u)	`indets(u)`	`Variables[u]`	`indets(u,PolyExpr)`

Figure 1. The polynomial operators in Maple, Mathematica, and MuPAD that are most similar to those in MPL. (Implementation: Maple (mws), Mathematica (nb), MuPAD (mnb).)