MATH REVIEW
for Textbook

Kinematics, Dynamics, and Design of Machinery

by

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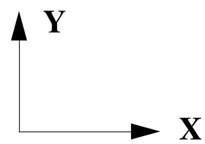
Math Review

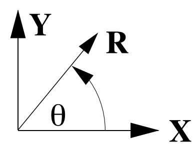
1 Coordinate Systems

Three types of coordinate systemsare commonly used in the study of mechanisms. These are:

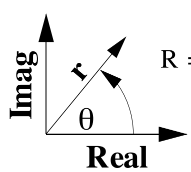
A) Cartesian



B) Polar



C) Complex



$R = x + \mathbf{j}y = re^{j\theta} = r(\cos\theta + \mathbf{j}\sin\theta)$

Through out this book, reference frame, coordinate frame, and coordinate system are used interchangeably.

Absolute reference frames are those which are fixed. If

$$\mathbf{F} = m\mathbf{a}$$

applies relative to the reference frame, it is called an inertial frame. It is not accelerating. For most terrestrial engineering problems, the earth can be taken as a fixed reference frame.

Absolute motion is motion relative to a fixed reference frame. Relative motion is motion relative to a reference frame that is moving with respect to the fixed reference frame. If the relative reference frame is not moving, it is treated as the same as the fixed reference frame.

We use coordinate systems to define both position quantities and angular quantities. Position quantities will typically refer to points and angular quantities will refer to bodies or coordinate systems.

2 Vectors

Vectorsare directed line segments. They have a direction and magnitude. They can be represented a number of different, equivalent ways.

In the discussion, the following definitions will be used (see Fig. 1):

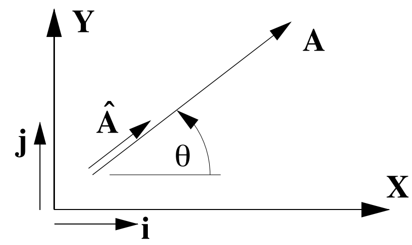


Fig. 1: Differrent ways to represent a vector.

$\mathbf{i}$ is a unit vector (1 unit long) in the direction of X
$\mathbf{j}$ is a unit vector (1 unit long) in the direction of Y
$\hat{\mathbf{A}}$ is a unit vector (1 unit long) in the direction of A
$\boldsymbol{\theta}$ is measured positive counterclockwise (CCW)
A is the magnitude of $\mathbf{A}$ and is a scalar.

Then,

$$\begin{aligned}
\mathbf{A} &= A\angle\theta \\
&= A(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) \\
&= A_x\mathbf{i} + A_y\mathbf{j} \\
&= A\hat{\mathbf{A}} \\
&= A(\hat{A}_x\mathbf{i} + \hat{A}_y\mathbf{j}) \\
&= Ae^{j\theta} \quad \text{(Note that here j is not the same as } \mathbf{j}\text{)}
\end{aligned}$$

Since each of these is equivalent,

$$A_x = A\cos\theta$$

$$A_y = A\sin\theta$$

$$A = \sqrt{A_x^2 + A_y^2}$$

$$\hat{A}_x = \cos\theta$$

$$\hat{A}_x = \cos\theta$$

$$\theta = \tan^{-1}\left(A_y / A_x\right)$$

3 Planar Vector Equations

Analytically, vectors are best handled in component form. Graphically, vectors are best handled in polar form.

Consider the vector equation

$$\mathbf{C = A + B} \tag{1}$$

This is the basic form of almost all of the vector equations that must be solved when analyzing linkages. Looking at this equation, we have a possibility of six variables which can be unknown:

$\mathbf{A} = A \angle \theta_A$
$\mathbf{B} = B \angle \theta_B$
$\mathbf{C} = C \angle \theta_C$

In one planar vector equation, we can have at most two unknowns, and the solution procedure varies depending on the variables which are unknown. We will look at two specific examples.

Example 1: (Solving Vector Equations with Two Unknown Magnitudes)

The values assumed for the first example are:

$$\mathbf{A} = 10 \angle 45° = 10\,(\cos 45\mathbf{i} + \sin 45\mathbf{j}) = 7.07\mathbf{i} + 7.07\mathbf{j}$$
$$\mathbf{B} = B \angle 0° = B\,(\cos 0\mathbf{i} + \sin 0\mathbf{j}) = B\mathbf{i}$$
$$\mathbf{C} = C \angle 30° = C\,(\cos 30\mathbf{i} + \sin 30\mathbf{j}) = C(\,0.867\mathbf{i} + 0.5\mathbf{j})$$

Note that in this example, we are assuming that one magnitude and all of the directions are known. We then need to determine the two unknown magnitudes B and C. The type of problem is typical of that encountered when solving velocity or acceleration problems.

Analytical Approach

For the analytical approach, convert the vectors in Eq. (1) into component form

$$\mathbf{C} = \mathbf{A} + \mathbf{B} \;\Rightarrow\; C_x\mathbf{i} + C_y\,\mathbf{j} = (A_x\mathbf{i} + A_y)\,\mathbf{j} + (B_x\mathbf{i} + B_y\,\mathbf{j})$$

or

$$\left.\begin{aligned} C_x &= A_x + B_x \\ C_y &= A_y + B_y \end{aligned}\right\} \text{ 2 scalar equations}$$

For this problem, the scalar equations are linear, and only one solution is possible. The solution to the equations is given by:

$$\begin{aligned} &C\,(0.867) = 7.07 + B_x \\ &C\,(0.5) = 7.07 \qquad \Rightarrow \qquad C = 14.14 \\ &\qquad\qquad\qquad\qquad\qquad\quad B = B_x = 5.19 \end{aligned}$$

Graphical Approach

To solve the equation graphically, start by picking a beginning location (pole) and a scale. Then follow the directions given by the equation. For each side of the equation, start at the pole and draw all of the known vectors first. For the last vector on each side of the equation, we do not know its magnitude, but we know its direction. Therefore, simply draw a line in the correct direction to represent the unknow vector. Each side of the equation will end with a vector of known direction but unknown magnitude. Each of these unknown vectors will be represented by a line. Where the lines from the two sides of the equation intersect is the solution to the problem. This is shown in Fig. 2

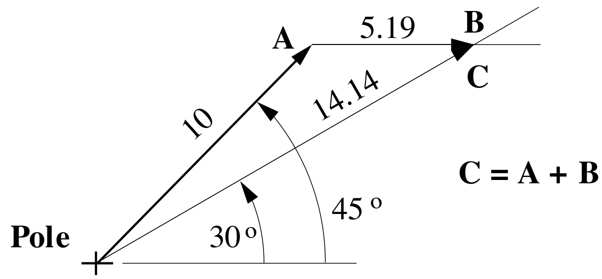


Fig. 2: Graphical solution to Eq. (1) when two unknown magnitudes are involved

Note that when we add vectors, we combine them **tail to head.** When we subtract vectors, we combine them **head to head**. This is shown in Fig.

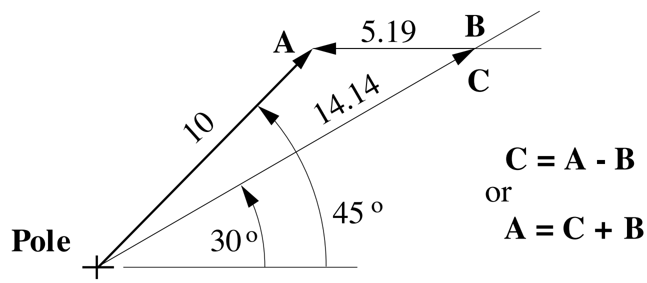


Fig. 3: Graphical solution to C = A - B

Example 2: (Solving Vector Equations with Two Unknown Directions)

In the second example, we will again solve Eq. (1) when there are two unknown directions. For this example, let the vectors be defined as follows:

$\mathbf{A} = 10 \angle 45° \quad = 7.07\mathbf{i} + 7.07\mathbf{j}$
$\mathbf{B} = 6 \angle \theta_B$
$\mathbf{C} = 15 \angle \theta_C$

<u>Graphical Approach</u>

We will solve the problem graphically first because this will reveal some insight into the type of solution that we should expect in the analytical approach. Again, we follow the directions given by the equations. The locus for the tip of a vector of unknown direction but known magnitude is represented initially by a circle as shown in Fig. A.4. Note that in general, the two circles can intersect at 0, 1, or 2 locations giving 0, 1, or 2 possible solutions. In this problem, two intersections and therefore two solutions are indicated.

<u>Analytical Approach</u>

For the analytical approach, start with the component equations.

$$\left.\begin{aligned} C_x = A_x + B_x &\Rightarrow 15\cos\theta_C = 7.07 + 6\cos\theta_B \\ C_y = A_y + B_y &\Rightarrow 15\sin\theta_C = 7.07 + 6\sin\theta_B \end{aligned}\right\} \qquad (2)$$

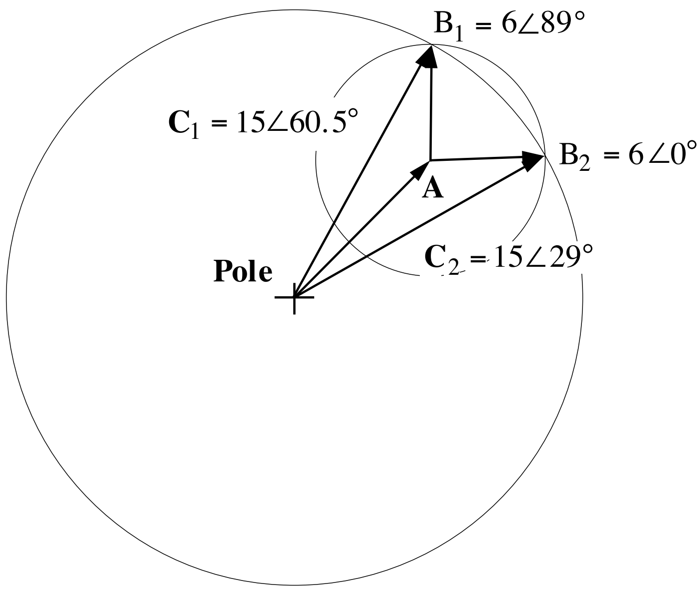


Fig. A.4: Solution to vector equation with unknown directions

To solve the scalar equations, we must decide which angle is to be found first. We will then isolate the term containing the other angle on the same side of the two equations, square both sides of both equations,and add and add the resulting two equations Because θ_C is already isolated on the left hand side of each equation, we will eliminate it and solve for θ_B. Squaring the equations gives:

$$225\cos^2\theta_C = 50 + 36\cos^2\theta_B + 84.85\cos\theta_B$$
$$225\sin^2\theta_C = 50 + 36\sin^2\theta_B + 84.85\sin\theta_B$$

Adding the two equations and using the identity $\sin^2\theta + \cos^2\theta = 1$, allows us to simplify the resulting equations as:

$$225 = 100 + 36 + 84.85(\cos\theta_B + \sin\theta_B)$$

or

$$1.0489 = \cos\theta_B + \sin\theta_B \qquad (3)$$

Now we can use the tangent of the half angle identities

$$\sin\theta = \frac{2\tan\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)}$$

and

$$\cos\theta = \frac{1-\tan^2\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)}$$

to convert Eq. (3). The resulting equation is:

$$1.0489 = \left[2\tan\left(\frac{\theta_B}{2}\right) + 1 - \tan^2\left(\frac{\theta_B}{2}\right)\right] \Big/ \left[1 + \tan^2\left(\frac{\theta_B}{2}\right)\right]$$

To simplify the equation, let $t = \tan\left(\frac{\theta_B}{2}\right)$, clear fractions, and collect terms. Then or

$$t^2 - 0.976t + 0.02385 = 0$$

Now we can use the quadratic formula $\left(x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right)$ to solve for t. The solutions are

$$t = \left[0.976 \pm \sqrt{(0.976)^2 - 0.09542}\right]/2 = \left[0.976 \pm 0.926\right]/2 = 0.250; 0.951$$

The two values for θ_B are then given by

$$\theta_B = 2\tan^{-1} t = 87.1°; 2.86°$$

Once the values for θ_B are known, we can use Eq. (2) to find θ_C. To do this, divide the second equation by the first to get

$$\frac{15\sin\theta_C}{15\cos\theta_C} = \tan\theta_C = \frac{7.07 + 6\sin\theta_B}{7.07 + 6\cos\theta_B} = \frac{7.07 + 6\sin 87.1°}{7.07 + 6\cos 87.1°}; \frac{7.07 + 6\sin 2.86°}{7.07 + 6\cos 2.86°} \tag{4}$$

or

$$\tan\theta_C = 1.77;\ 0.584$$

and

$$\theta_C = 60.5°;\ 30.3°.$$

Note that in general, we must use the inverse tangent function rather than the inverse sine or inverse cosine functions to find a given angle because in inverse sign and cosine functions do not give unambigiously the proper quadrants for the angle. When we use the inverse tangent function, we must preserve the signs on the numerator and denominator in Eq. (5) in order to establish the proper quadrant for the unknown angle..

4 Vector Cross Products

Cross products are used to compute velocities ($\boldsymbol{\omega}$ **x r**), accelerations ($\boldsymbol{\alpha}$ **x r**) and torques (**rxF**). To illustrate the computation of the cross produce, consider two vectors (**A**, **B**) which can be represented symbolically by:

$$\mathbf{A} = A_x\,\mathbf{i} + A_y\,\mathbf{j} + A_z\,\mathbf{k}$$
$$\mathbf{B} = B_x\,\mathbf{i} + B_y\,\mathbf{j} + B_z\,\mathbf{k} \tag{5}$$

The cross product, **A x B**, can be computed in at least three different ways:

<u>Direct Approach</u>

The direct approach is useful when solving problems graphically and when planar problems are involved. For this approach, we compute the magnitude of the cross product from

$$|\mathbf{A} \times \mathbf{B}| = |\mathbf{A}||\mathbf{B}|\sin\theta$$

And determine the direction of the resulting vector using the right hand screw rule as illustrated in Fig. A. 5.

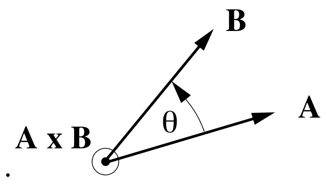


Fig. 5: Right hand rule to determine the direction of a cross product

Determinant Approach

The determinate approach is useful when computing the cross product analytically. In this approach we develop a determinant with the unit vectors for the first row, the components of the first vector for the second row, and the components of the second vector for the third row. The expansion of the resulting determinant is the cross product. This is illustrated in the following:

$$\mathbf{AxB} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \mathbf{i}(A_yB_z) + \mathbf{j}(A_zB_x) + \mathbf{k}(A_xB_y) - \mathbf{i}(A_zB_y) - \mathbf{j}(A_xB_z) - \mathbf{k}(A_yB_x)$$
$$= (A_yB_z - A_zB_y)\mathbf{i} + (A_zB_x - A_xB_z)\mathbf{j} + (A_xB_y - A_yB_x)\mathbf{k}$$

This approach works for two and three component vectors which are the types that we deal with in linkage analysis. For more general vectors with more than three components, the product approach must be used.

Product Approach

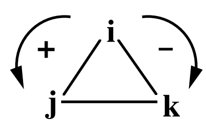
In the product approach, we represent the vectors in component form, and perform the cross product term by term. This is shown in the following.

$$\begin{aligned} \mathbf{AxB} &= (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k})\mathbf{x}(B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}) = A_xB_x(\mathbf{ixi}) + A_xB_y(\mathbf{ixj}) + A_xB_z(\mathbf{ixk}) \\ &= A_yB_x(\mathbf{jxi}) + A_yB_y(\mathbf{jxj}) + A_yB_z(\mathbf{jxk}) = A_zB_x(\mathbf{kxi}) + A_zB_y(\mathbf{kxj}) + A_zB_z(\mathbf{kxk}) \end{aligned} \tag{6}$$

To simplify the resulting expression, note that **i**, **j**, and **k** are mutually orthogonal. Therefore,

i x i = j x j = k x k = 0
i x j = k; **j x k = i**; **k x i = j**
j x i = -k; **k x j = -i**; **i x k = -j**

We can remember the order using the following figure.



Then, Eq. (6) can be simplified to the following:

$$\mathbf{AxB} = (A_yB_z - A_zB_y)\mathbf{i} + (A_zB_x - A_xB_z)\mathbf{j} + (A_xB_y - A_yB_x)\mathbf{k}$$

5 Dot (Vector, Scalar, Inner) Products

The dot product is used when computing work terms and also when looking for the component of a vector in a given direction. Unlike the cross product, the dot product is a scalar. To illustrate the computation of the dot produce, again consider two vectors ($\mathbf{A}$, $\mathbf{B}$) which can be represented symbolically by Eq. (5).

$\mathbf{A} = A_x\,\mathbf{i} + A_y\,\mathbf{j} + A_z\,\mathbf{k}$
$\mathbf{B} = B_x\,\mathbf{i} + B_y\,\mathbf{j} + B_z\,\mathbf{k}$

The dot product, $\mathbf{A} \cdot \mathbf{B}$, can be computed in two convenient ways:

<u>Direct Approach</u>

The direct approach is again useful when solving problems graphically and when planar problems are involved. For this approach, we compute the dot product from

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}|\cos\theta \tag{7}$$

where θ is the angle between the two vectors. From this general definition, it is obvious that

$$\begin{gathered}\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1 \\ \mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{i} = \mathbf{k} \cdot \mathbf{j} = \mathbf{i} \cdot \mathbf{k} = 0\end{gathered} \tag{8}$$

<u>Product Approach</u>

In the product approach, we perform the dot product term by term and use Eq. (8) to simplify the results. The final expression is

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z \tag{9}$$

6 Angle between two vectors

To locate the angle between two vectors, we can use Eqs. (7) and (9). From Eq. (7),

$$\cos\theta = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}||\mathbf{B}|}$$

We can use Eq. (9) to compute the $\mathbf{A} \cdot \mathbf{B}$. The vector magnitudes can be computed from

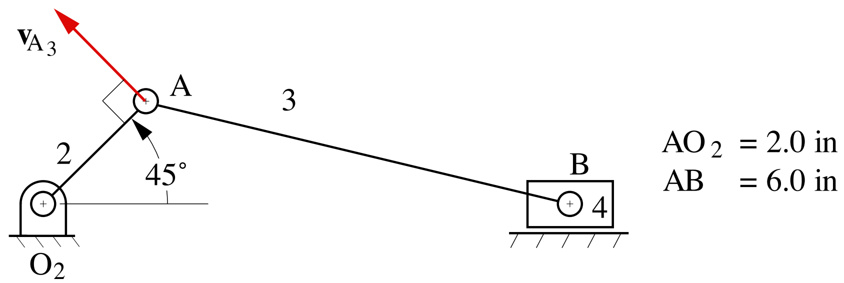
$$|\mathbf{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2} \text{ and } |\mathbf{B}| = \sqrt{B_x^2 + B_y^2 + B_z^2} .$$

7 Math Review Exercise Problems

Exercise 1

The slider-crank mechanism has the velocity of Point A_3 given by the vector shown. The velocity $\mathbf{v}_{B_3/A_3}$ is perpendicular to AB, and the velocity of B_3 is horizontal. The mechanism is drawn full scale, and the magnitude of $\mathbf{v}_{A_3}$ is 10 in/sec. Graphically determine $\mathbf{v}_{B_3}$ if the vectors are related by the equation

$$\mathbf{v}_{B_3} = \mathbf{v}_{A_3} + \mathbf{v}_{B_3/A_3}$$



Exercise 2

Given the two vectors: $\mathbf{A} = 5\mathbf{i} + 3\mathbf{j}$ and $\mathbf{B} = 4\mathbf{i} - 7\mathbf{j}$, do the following both analytically and graphically:

a) Find **A + B**, **A - B,** and **A x B.**
b) Find the magnitude of **A + B**
c) Find the angle that **A + B** makes with the X axis

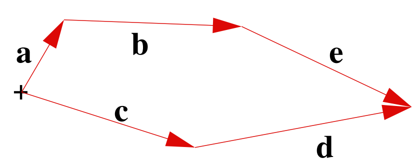
Exercise 3

Find the values of B and C both graphically and analytically given the equation **A + B = C** and:

$\mathbf{A} = 5 \angle 53.13^\circ$; $\mathbf{B} = B \angle -50^\circ$; $\mathbf{C} = C \angle 0^\circ$

Exercise 4

Write the vector equation represented by the figure.



Exercise 5

Find θ_b and θ_c both analytically and graphically given the vector equation **A = B + C** and:

$\mathbf{A} = 10 \angle 60^\circ$; $\mathbf{B} = 8 \angle \theta_b$; $\mathbf{C} = 10 \angle \theta_c$

Exercise 6
Resolve Exercise 5 when $\mathbf{A} = 20 \angle 45^\circ$.

Exercise 7
Find θ_b and θ_c <u>both</u> analytically and graphically given the vector equation $\mathbf{A = B + C}$ and:

$\mathbf{A} = 20 \angle 45^\circ$; $\mathbf{B} = 15 \angle \theta_b$; $\mathbf{C} = 10 \angle \theta_c$

Exercise 8
Resolve Exercise 7 if $\mathbf{A} = 28 \angle 60^\circ$.

Exercise 9
Find θ_b and θ_c <u>both</u> analytically and graphically given the vector equation $\mathbf{A = B + C}$ and:
$\mathbf{A} = 15 \angle 30^\circ$; $\mathbf{B} = 10 \angle \theta_b$; $\mathbf{C} = 10 \angle \theta_c$

Exercise 10
Find θ_b and θ_c <u>both</u> analytically and graphically given the vector equation $\mathbf{A = B + C}$ and:
$\mathbf{A} = 10 \angle 60^\circ$; $\mathbf{B} = 5 \angle \theta_b$; $\mathbf{C} = 8 \angle \theta_c$

Exercise 11
Resolve Exercise 16 if $\mathbf{A} = 20 \angle 45^\circ$. Interpret the analytical solution based on your graphical solution.

Exercise 12

Given: $\mathbf{P_1} = 5\mathbf{i} + \mathbf{j}$; $\mathbf{P_2} = \mathbf{i} + 4\mathbf{j} + \mathbf{k}$; $\mathbf{P_3} = 3\mathbf{i} + 8\mathbf{j} + 4\mathbf{k}$, find:

a) $\mathbf{P_1} + \mathbf{P_2} + \mathbf{P_3}$
b) $(\mathbf{P_1} + \mathbf{P_2}) \bullet \mathbf{P_3}$
c) Angle between $\mathbf{P_1}$ and $\mathbf{P_2}$
d) $(\mathbf{P_1} \times \mathbf{P_2}) \times \mathbf{P_3}$
e) If $\mathbf{P_1} = 2 \angle\theta + b\angle 45^\circ$, what are θ and b. (Do both analytically and graphically)

Exercise 13

Given: $\mathbf{P_1} = 3\mathbf{i} + \mathbf{j}$; $\mathbf{P_2} = \mathbf{i} + 3\mathbf{j} + \mathbf{k}$; $\mathbf{P_3} = 5\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}$, find:

a) $\mathbf{P_1} + \mathbf{P_2} + \mathbf{P_3}$
b) $(\mathbf{P_1} + \mathbf{P_2}) \bullet \mathbf{P_3}$
c) Angle between $\mathbf{P_1}$ and $\mathbf{P_2}$
d) $(\mathbf{P_1} \times \mathbf{P_2}) \times \mathbf{P_3}$
e) If $\mathbf{P_1} = 2 \angle\theta + b\angle 45^\circ$, what are θ and b. (Do both analytically and graphically)

Exercise 14

Given: $\mathbf{P_1} = 5 + 2\mathbf{j}$; $\mathbf{P_2} = 2\mathbf{i} + 6\mathbf{j} + \mathbf{k}$; $\mathbf{P_3} = 3\mathbf{i} + 6\mathbf{j} + 5\mathbf{k}$, find:

a) $\mathbf{P_1} + \mathbf{P_2} + \mathbf{P_3}$

b) $(\mathbf{P_1} + \mathbf{P_2}) \bullet \mathbf{P_3}$

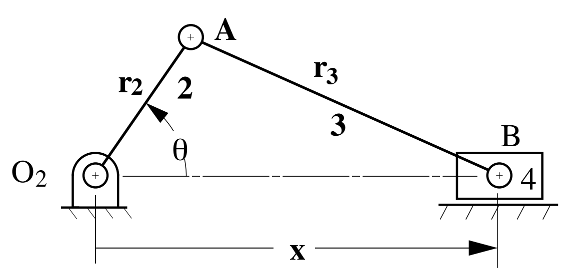
c) Angle between $\mathbf{P_1}$ and $\mathbf{P_2}$

d) $(\mathbf{P_1} \times \mathbf{P_2}) \times \mathbf{P_3}$

e) If $\mathbf{P_1} = 5\angle\theta + b\angle 45°$, what are the values for θ and b that satisfy the equation? (Do both analytically and graphically)

Exercise 15

Consider the slider-crank mechanism given below.



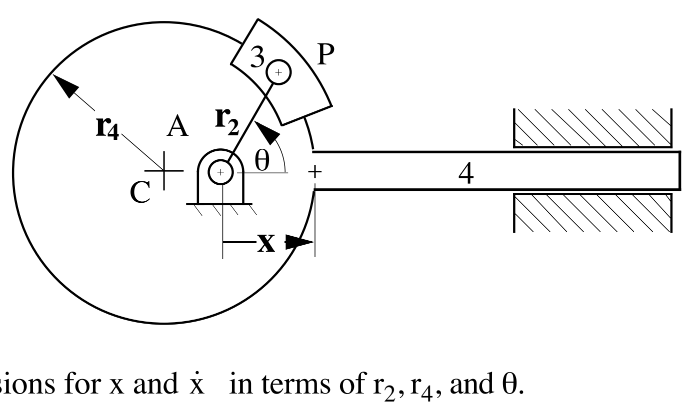
a) Find an expression for x in terms of r_2, r_3, and θ.

b) What are the limits on x?

c) Plot $\dot{x}$ vs θ, using $r_2 = 1.0$ m, $r_3 = 3.0$ m, and $\dot{\theta} = 1.0$ rad/sec.

Exercise 16

Consider the mechanism given below.



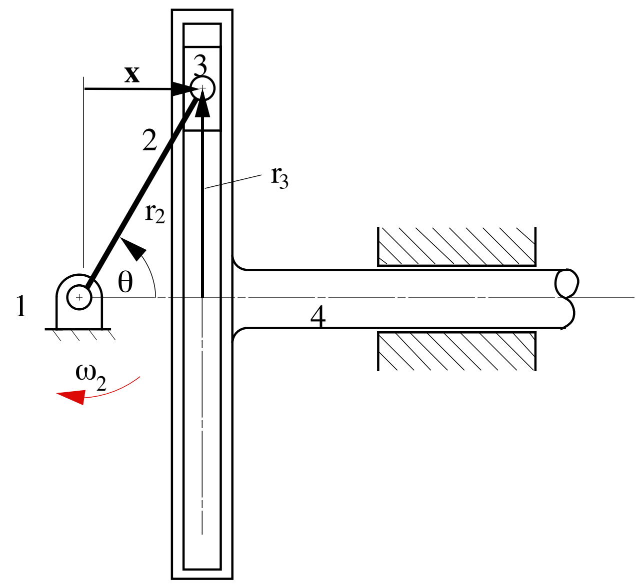
a) Find an expressions for x and $\dot{x}$ in terms of r_2, r_4, and θ.

b) What are the limits on x?

c) Plot $\dot{x}$ vs θ, using $r_2 = 3.0$ m, $r_4 = 4.0$ m, and $\dot{\theta} = 1.0$ rad/sec.

Exercise 17

Consider the Scotch-Yoke mechanism given below.



a) Find an expression for x in terms of r_2 and θ.

b) What are the limits on x?

c) Plot $\dot{x}$ vs θ, using $r_2 = 1.0$ m and $\dot{\theta} = 1.0$ rad/sec.

d) How would you characterize the motion of the Scotch-Yoke Mechanism?