Fourier Series

REPRESENTATION OF A VECTOR USING ANOTHER VECTOR

Consider two vectors v_1 and v_2 as shown and let θ be the smallest angle between the two vectors.

$$v_1 = C_{12}v_2 + v_e$$
$$v_e = v_1 - C_{12}v_2$$

where v_e is error vector

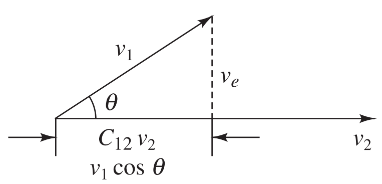


Fig. 1

We can approximate the vector v_1 using vector v_2 by drawing a perpendicular from the end of v_1 onto v_2. The value of C_{12} is selected such that error is minimum. Consider the following two cases;

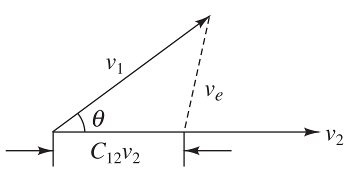


Fig. 2

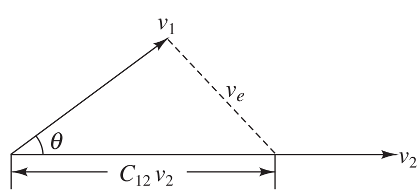


Fig. 3

From these figures, it is clear that error is minimum in Fig. 1. Hence, in that case we can approximate $v_1 \cong C_{12}\, v_2$.

Also,
$$\begin{aligned} v_1 \cdot v_2 &= \text{magnitude of } v_2\text{. projection of } v_1 \text{ on } v_2 \\ &= v_2 \cdot v_1 \cos\theta \\ &= v_2 \cdot C_{12}\, v_2 \\ &= C_{12}\, v_2^2 \end{aligned}$$

$$C_{12} = \frac{v_1 \cdot v_2}{v_2^2}$$

In case of mutually orthogonal vectors $\theta = 90°$,

$$v_1 \cdot v_2 = 0$$
$$C_{12} = 0.$$

Then we cannot express v_1 in terms of v_2. But we can express any other vector in terms of these two vectors.

Hence, condition for orthogonality is $v_1 \cdot v_2 = 0$.

Similarly we can express a signal in terms of another signal.

ORTHOGONALITY OF SIGNALS

Consider two signals $f_1(t)$ and $f_2(t)$. We want to approximate $f_1(t)$ in terms of $f_2(t)$ over a certain interval $(t_1 < t < t_2)$.

$$f_1(t) \cong C_{12} f_2(t) \qquad \text{for } t_1 < t < t_2$$

The error in this approximation will be

$$f_e(t) = f_1(t) - C_{12} f_2(t)$$

C_{12} is selected such that error between the actual function and the approximated function is minimum. For minimising the error $f_e(t)$ over the interval t_1 to t_2, we have to minimise the average (or mean) of the square of error function $f_e(t)$.

The mean square error ε is given by,

$$\varepsilon = \frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} f_e^2(t)\,dt$$

$$= \frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} [f_1(t) - C_{12} f_2(t)]^2\,dt$$

$$= \frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} [f_1^2(t) - 2C_{12} f_1(t) f_2(t) + C_{12}^2 f_2^2(t)]dt$$

$$= \frac{1}{(t_2 - t_1)} \left[\int_{t_1}^{t_2} f_1^2(t)dt - 2C_{12}\int_{t_1}^{t_2} f_1(t) f_2(t)dt + C_{12}^2 \int_{t_1}^{t_2} f_2^2(t)]dt\right]$$

To find the value of C_{12} which will minimise ε we must have

$$\frac{\partial \varepsilon}{\partial C_{12}} = 0$$

i.e. $$\frac{\partial \varepsilon}{\partial C_{12}} = \frac{\partial}{\partial C_{12}}\left\{\frac{1}{(t_2 - t_1)}\left[\int_{t_1}^{t_2} f_1^2(t)dt - 2C_{12}\int_{t_1}^{t_2} f_1(t) f_2(t)dt + C_{12}^2\int_{t_1}^{t_2} f_2^2(t)]dt\right]\right\}$$

Changing the order of integration and differentiation, we get,

$$\frac{\partial \varepsilon}{\partial C_{12}} = \frac{1}{(t_2 - t_1)}\left[\int_{t_1}^{t_2} \frac{\partial}{\partial C_{12}} f_1^2(t)dt - 2\int_{t_1}^{t_2} f_1(t) f_2(t)dt + 2C_{12}\int_{t_1}^{t_2} f_2^2(t)]dt\right]$$

$$= \frac{1}{(t_2 - t_1)}\left[0 - 2\int_{t_1}^{t_2} f_1(t) f_2(t)dt + 2C_{12}\int_{t_1}^{t_2} f_2^2(t)]dt\right]$$

Now, when $\frac{\partial \varepsilon}{\partial C_{12}} = 0$ we get

$$\int_{t_1}^{t_2} f_1(t) f_2(t)dt = C_{12}\int_{t_1}^{t_2} f_2^2(t)dt$$

$$C_{12} = \frac{\int_{t_1}^{t_2} f_1(t) f_2(t)dt}{\int_{t_1}^{t_2} f_2^2(t)dt}$$

When C_{12} is zero, then the signal $f_1(t)$ contains no component of signal $f_2(t)$ and the two signals are said to be orthogonal over the interval (t_1, t_2). Thus the two functions are orthogonal over an interval (t_1, t_2) if

$$\int_{t_1}^{t_2} f_1(t) f_2(t)dt = 0$$

If in addition, $\int_{t_1}^{t_2} | f_1(t) |^2 dt = 1 = \int_{t_1}^{t_2} | f_2(t) |^2 dt$, the functions are said to be normalized and hence are called orthonormal.

Note: (1) If a set of signals is not orthonormal, we can converts the orthogonal set to orthonormal by dividing $f_n(t)$ by $\sqrt{K_n}$.

Note: (2) If $f(t)$ is a complex function of real variable t,

$$\int_{t_1}^{t_2} f_1(t) f_2^*(t)dt = 0$$

and $$\int_{t_1}^{t_2} f_1(t)^2 dt = K$$

Example 1 *Show that the functions sin n $\omega_0 t$ and sin m $\omega_0 t$ are orthogonal over any interval* $\left(t_0, t_0 + \frac{2\pi}{\omega_0}\right)$.

Soluton Let

$$f_1(t) = \sin n\omega_0 t$$
$$f_2(t) = \sin m\omega_0 t$$

$$\int_{t_1}^{t_2} f_1(t) f_2(t) dt = \int_{t_0}^{t_0+\frac{2\pi}{\omega_0}} \sin n\omega_0 t \sin \omega_0 t \, dt$$

$$= \int_{t_0}^{t_0+\frac{2\pi}{\omega_0}} \frac{1}{2}[\cos(n-m)\omega_0 t - \cos(n+m)\omega_0 t]dt$$

$$= \frac{1}{2\omega_0}\left[\frac{1}{(n-m)}\sin(n-m)\omega_0 t - \frac{1}{(n+m)}\sin(n+m)\omega_0 t\right]_{t_0}^{t_0+\frac{2\pi}{\omega_0}}$$

$$= \frac{1}{2\omega_0}\left[\frac{1}{(n-m)}\sin(n-m)\omega_0\left(t_0 + \frac{2\pi}{\omega_0}\right) - \frac{1}{(n+m)}\sin(n+m)\omega_0\left(t_0 + \frac{2\pi}{\omega_0}\right) - \frac{1}{(n-m)}\sin(n-m)\omega_0 t_0 + \frac{1}{(n+m)}\sin(n+m)\omega_0 t_0\right]$$

Since n and m are integers, $(n - m)$ and $(n + m)$ are also integers.

$$\int_{t_0}^{t_0+\frac{2\pi}{\omega_0}} \sin n\omega_0 t \sin m\omega_0 t \, dt = 0$$

Hence two functions are orthogonal.

Example 2 *Show that cos $n\omega_0 t$ and cos $m\omega_0 t$ are orthogonal over the interval* $\left(t_0, t_0 + \frac{2\pi}{\omega_0}\right)$.

Solution

$$\int_{t_1}^{t_2} f_1(t) f_2(t) dt = \int_{t_0}^{t_0+\frac{2\pi}{\omega_0}} \cos n\omega_0 t \cos m\omega_0 t \, dt$$

$$= \int_{t_0}^{t_0+\frac{2\pi}{\omega_0}} \frac{1}{2}[\cos(n+m)\omega_0 t + \cos(n-m)\omega_0 t]dt$$

$$= \frac{1}{2\omega_0}\left[\frac{1}{(n+m)}\sin(n+m)\omega_0 t + \frac{1}{(n-m)}\sin(n-m)\omega_0 t\right]_{t_0}^{t_0+\frac{2\pi}{\omega_0}}$$

$$= \frac{1}{2\omega_0}\left[\frac{1}{(n+m)}\sin(n+m)\omega_0\left(t_0+\frac{2\pi}{\omega_0}\right) - \frac{1}{(n+m)}\sin(n+m)\omega_0 t_0\right.$$

$$\left. + \frac{1}{(n-m)}\sin(n-m)\omega_0\left(t_0+\frac{2\pi}{\omega_0}\right)t_0 - \frac{1}{(n+m)}\sin(n-m)\omega_0 t_0\right]$$

Since n and m are integers, $(n + m)$ and $(n - m)$ are also integers.

$$\therefore \qquad \int_{t_0}^{t_0+\frac{2\pi}{\omega_0}} \frac{1}{2}\cos n\omega_0 t \cdot \cos m\omega_0 t dt = 0$$

Hence, two functions are orthogonal.

Example 3 *Show that* $\sin n\omega_0 t$ *and* $\cos m\omega_0 t$ *are orthogonal over the interval* $\left(t_0, t_0+\frac{2\pi}{\omega_0}\right)$.

Solution

$$\int_{t_0}^{t_0+\frac{2\pi}{\omega_0}} \sin n\omega_0 t \cos m\omega_0 t dt$$

$$= \int_{t_0}^{t_0+\frac{2\pi}{\omega_0}} \frac{1}{2}\left[\sin(n+m)\omega_0 t + \sin(n-m)\omega_0 t\right] dt$$

$$= \frac{1}{2\omega_0}\left[\frac{-1}{(n+m)}\cos(n+m)\omega_0 t - \frac{1}{(n-m)}\cos(n-m)\omega_0 t\right]_{t_0}^{t_0+\frac{2\pi}{\omega_0}}$$

$$= \frac{1}{2\omega_0}\left[-\frac{1}{(n+m)}\cos(n+m)\omega_0\left(t_0+\frac{2\pi}{\omega_0}\right) + \frac{1}{(n+m)}\cos(n+m)\omega_0 t_0\right.$$

$$\left. - \frac{1}{(n-m)}\cos(n-m)\omega_0\left(t_0+\frac{2\pi}{\omega_0}\right)t_0 + \frac{1}{(n-m)}\cos(n-m)\omega_0 t_0\right]$$

Since n and m are integers, $(n + m)$ and $(n - m)$ are also integers.

$$\int_{t_0}^{t_0+\frac{2\pi}{\omega_0}} \sin n\omega_0 t \cos m\omega_0 t\, dt = 0$$

Hence two functions are orthogonal.

Example 4 *A rectangular function f(t) is given by*

$$f(t) = 1 \; (0 < t < \pi)$$
$$= -1 \; (\pi < t < 2\pi)$$

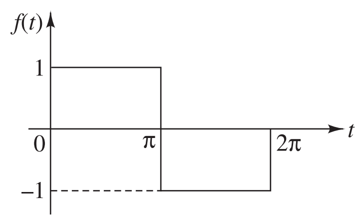


Fig. 4

Approximate this function by a waveform sin t over the interval (0, 2π) such that mean square error is minimum.

Solution The function $f(t)$ is approximated over the interval (0, 2π) as

$$f(t) = C_{12} \sin t$$

$$C_{12} = \frac{\int_0^{2\pi} f(t) \sin t\,dt}{\int_0^{2\pi} \sin^2 t\,dt}$$

$$= \frac{\int_0^{\pi} \sin t\,dt + \int_{\pi}^{2\pi} -\sin t\,dt}{\int_0^{2\pi} \frac{1}{2}(1 - \cos 2t)dt}$$

$$= \frac{[-\cos t]_0^{\pi} + [\cos t]_{\pi}^{2\pi}}{\frac{1}{2}\left[t - \frac{\sin 2t}{2}\right]_0^{2\pi}}$$

$$= \frac{-\cos\pi + \cos 0 + \cos 2\pi - \cos\pi}{\frac{1}{2}\left[2\pi - \frac{\sin 4\pi}{2} + \frac{\sin 0}{2}\right]}$$

$$= \frac{4}{\frac{1}{2}(2\pi)} = \frac{4}{\pi}$$

$$f(t) = \frac{4}{\pi} \sin t$$

Example 5 *A rectangular function f(t) is approximated by sine function show that the error in the approximation is orthogonal to sin t in the interval (0, 2π).*

Solution The error in the approximation is

$$f_e(t) = f(t) - \frac{4}{\pi} \sin t$$

$$\int_0^{2\pi} f_e(t) \sin t\,dt = \int_0^{2\pi} \left[f(t) - \frac{4}{\pi} \sin t \right] \sin t\,dt$$

$$= \int_0^{2\pi} f(t) \sin t\,dt - \frac{4}{\pi} \int_0^{2\pi} \sin^2 t\,dt$$

$$= \int_0^{\pi} \sin t\,dt + \int_0^{2\pi} -\sin t\,dt - \frac{4}{\pi} \int_0^{2\pi} \frac{1}{2} (1 - \cos 2t)dt$$

$$= [-\cos t]_0^{\pi} + [\cos t]_{\pi}^{2\pi} - \frac{2}{\pi} \left[t - \frac{\sin 2t}{2} \right]_0^{2\pi}$$

$$= 2 + 2 - 4$$

$$= 0$$

Hence the error function $f_e(t)$ is orthogonal to sin t.

Example 6 Show that exponential functions $\{e^{jn\omega_0 t}\}$, $n = 0, \pm 1, \pm 2, \ldots$) is orthogonal over an interval $\left(t_0, t_0 + \frac{2\pi}{\omega_0} \right)$.

Solution

$$I = \int_{t_0}^{t_0 + \frac{2\pi}{\omega_0}} (e^{jn\omega_0 t})^* dt$$

$$= \int_{t_0}^{t_0 + \frac{2\pi}{\omega_0}} e^{jn\omega_0 t} e^{-jm\omega_0 t} dt$$

$$= \int_{t_0}^{t_0 + \frac{2\pi}{\omega_0}} dt = \frac{2\pi}{\omega_0} \qquad \text{when } n = m$$

$$I = \frac{1}{j(n-m)\omega_0} \left[e^{j(n-m)\omega_0 t} \right]_{t_0}^{t_0 + \frac{2\pi}{\omega_0}}$$

$$= \frac{1}{j(n-m)\omega_0} e^{j(n-m)\omega_0 t_0} \left[e^{j2\pi(n-m)} - 1 \right] (\because e^{j2\pi(n-m)} = 1)$$

$$= 0$$

Thus, $\{e^{jn\omega_0 t}\}$, $n = 0, \pm 1, \pm 2, \ldots$) is orthogonal over the interval $\left(t_0, t_0 + \frac{2\pi}{\omega_0}\right)$.

ORTHOGONAL SIGNAL SPACE

Any function $f(t)$ can be expressed as a sum of its components along a set of mutually orthogonal functions if these functions form a complete set.

Approximation of a Function by a Set of Mutually Orthogonal Functions

Consider a set of n mutually orthogonal functions $f_1(t), f_2(t), \ldots f_n(t)$ over an interval (t_1, t_2) i.e.

$$\int_{t_1}^{t_2} f_i(t)\, f_j(t) dt = 0 \qquad i \neq j$$

$$\int_{t_1}^{t_2} f_j^2(t) dt = k_j \qquad i = j$$

where k_j is a constant.

Let the function $f(t)$ be approximated over an interval (t_1, t_2) by a linear combination of these n mutually orthogonal functions.

$$f(t) = C_1 f_1(t) + C_2 f_2(t) + \ldots + C_n f_n(t)$$

$$= \sum_{j=1}^{n} C_j f_j(t) \qquad t_1 < t < t_2$$

The error in this approximation is

$$f_e(t) = f(t) - \sum_{j=1}^{n} C_j f_j(t)$$

and mean square error is

$$\varepsilon = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} f_e^2(t) dt$$

$$= \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \left[f(t) - \sum_{j=1}^{n} C_j f_j(t) \right]^2 dt$$

$$= \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \left[f^2(t) - 2f(t) \sum_{j=1}^{n} C_j f_j(t) + \sum_{j=1}^{n} C_j^2 f_j^2(t) \right] dt$$

Thus ε is a function of $C_1, C_2, \ldots C_n$ and to minimise ε, we must have

$$\frac{\partial \varepsilon}{\partial C_1} = \frac{\partial \varepsilon}{\partial C_2} = \ldots = \frac{\partial \varepsilon}{\partial C_n} = 0$$

$$\frac{\partial \varepsilon}{\partial C_j} = \frac{1}{t_2 - t_1}\left[\int_{t_1}^{t_2} -2f(t)f_j(t)dt + 2C_j\int_{t_1}^{t_2} f_j^2(t)dt\right] = 0$$

$$\int_{t_1}^{t_2} f(t)f_j(t)dt = C_j\int_{t_1}^{t_2} f_j^2(t)dt$$

$$C_j = \frac{\int_{t_1}^{t_2} f(t)f_j(t)dt}{\int_{t_1}^{t_2} f_j^2(t)dt}$$

$$= \frac{1}{k_j}\int_{t_1}^{t_2} f(t)f_j(t)dt$$

Hence, when a set of n functions $f_1(t), f_2(t) \ldots f_n(t)$ mutually orthogonal over the interval (t_1, t_2) is given, it is possible to approximate the function $f(t)$ over this interval by a linear combination of these n functions.

$$f(t) = C_1 f_1(t) + C_2 f_2(t) + \ldots + C_n f_n(t)$$

$$= \sum_{j=1}^{n} C_j f_j(t)$$

Evaluation of mean square error:

$$\varepsilon = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} f_e^2(t)dt$$

$$= \frac{1}{t_2 - t_1}\int_{t_1}^{t_2}\left[f(t) - \sum_{j=1}^{n} C_j f_j(t)\right]^2 dt$$

$$= \frac{1}{t_2 - t_1}\left[\int_{t_1}^{t_2} f^2(t)dt + \sum_{j=1}^{n} C_j^2\int_{t_1}^{t_2} f_j^2(t)dt - 2\sum_{j=1}^{n} C_j\int_{t_1}^{t_2} f(t)f_j(t)dt\right]$$

By definition,

$$C_j = \frac{1}{K_j}\int_{t_1}^{t_2} f(t)f_j(t)dt$$

$$C_j K_j = \int_{t_1}^{t_2} f(t)f_j(t)dt$$

$$\varepsilon = \frac{1}{t_2 - t_1}\left[\int_{t_1}^{t_2} f^2(t)dt - 2\sum_{j=1}^{n} C_j^2 K_j + \sum_{j=1}^{n} C_j^2 K_j\right]$$

$$= \frac{1}{t_2 - t_1}\left[\int_{t_1}^{t_2} f^2(t)dt - \sum_{j=1}^{n} C_j^2 K_j\right]$$

$$= \frac{1}{t_2 - t_1}\left[\int_{t_1}^{t_2} f^2(t)dt - (C_1^2 K_1 + C_2^2 K_2 + \cdots + C_n^2 K_n)\right]$$

Example 7 *A signal* $f(t) = 1 \quad 0 < t < \pi$

$= -1 \quad \pi < t < 2\pi$

is approximated by the signal sin nt. Find the error in approximation when n = 1, 2, 3, …

Solution

$$C_1 = \frac{1}{K_1}\int_{t_1}^{t_2} f(t)\sin nt\,dt$$

$$= \frac{1}{K_1}\left[\int_0^{\pi} \sin t\,dt + \int_{\pi}^{2\pi} (-1)\sin t\,dt\right]$$

$$= \frac{1}{K_1}\left[(-\cos t)_0^{\pi} + (\cos t)_{\pi}^{2\pi}\right]$$

$$= \frac{1}{K_1}[-\cos\pi + \cos 0 + \cos 2\pi - \cos\pi]$$

$$= \frac{1}{K_1}[1 + 1 + 1 + 1] = \frac{4}{K_1}$$

$$K_1 = \int_{t_1}^{t_2} \sin^2 t\,dt = \int_0^{2\pi} \sin^2 t\,dt$$

$$= \int_0^{2\pi}\left(\frac{1 - \cos 2t}{2}\right)dt$$

$$= \frac{1}{2}\left[t - \frac{\sin 2t}{2}\right]_0^{2\pi} = \frac{1}{2}(2\pi) = \pi$$

$$C_1 = \frac{4}{\pi}$$

$$C_2 = \frac{1}{K_2}\left[\int_0^{\pi} \sin 2t\,dt + \int_{\pi}^{2\pi} (-1)\sin 2t\,dt\right]$$

$$= \frac{1}{K_2}\left[\left(\frac{-\cos 2t}{2}\right)_0^{\pi} + \left(\frac{\cos 2t}{2}\right)_{\pi}^{2\pi}\right]$$

$$= \frac{1}{K_2}\cdot\frac{1}{2}[-\cos 2\pi + \cos 0 + \cos 4\pi - \cos 2\pi]$$

$$= \frac{1}{K_2} \cdot \frac{1}{2}\ [-1 + 1 + 1 - 1]$$

$$= 0$$

$$C_3 = \frac{1}{K_3}\left[\int_0^{\pi} \sin 3t\,dt + \int_{\pi}^{2\pi} (-1)\sin 3t\right]$$

$$= \frac{1}{K_3}\left[\left(\frac{-\cos 3t}{3}\right)_0^{\pi} + \left(\frac{\cos 3t}{3}\right)_{\pi}^{2\pi}\right]$$

$$= \frac{1}{K_3} \cdot \frac{1}{3}\ [-\cos 3\pi + \cos 0 + \cos 6\pi - \cos 3\pi]$$

$$= \frac{1}{K_3} \cdot \frac{1}{3}\ [1 + 1 + 1 + 1] = \frac{4}{3K_3}$$

$$K_3 = \int_0^{2\pi} \sin^2 3t\,dt$$

$$= \int_0^{2\pi} \left(\frac{1 - \cos 6t}{2}\right) dt = \frac{1}{2}\left[t - \frac{\sin 6t}{6}\right]_0^{2\pi}$$

$$= \frac{1}{2}\left[2\pi - \frac{\sin 12\pi}{6} + \sin 0\right]$$

$$= \pi$$

Error $$\varepsilon = \frac{1}{2\pi}\left[\int_0^{2\pi} f^2(t)dt - \left\{\left(\frac{4}{\pi}\right)^2 \pi + 0 + \left(\frac{4}{3\pi}\right)^2 \cdot \pi\right\}\right]$$

$$= \frac{1}{2\pi}\left[\int_0^{\pi} 1^2\,dt + \int_{\pi}^{2\pi} (-1)^2\,dt - \frac{16}{\pi} - \frac{16}{9\pi}\right]$$

$$= \frac{1}{2\pi}\left[\pi - 0 + 2\pi - \pi - \frac{16}{\pi} - \frac{16}{9\pi}\right]$$

$$= 1 - \frac{80}{9\pi^2} = 0.099 \text{ J.}$$

or $$C_r = \frac{\int_0^{2\pi} f(t)\sin rt\,dt}{\int_0^{2\pi} \sin^2 rt\,dt}$$

Now
$$K_r = \int_0^{2\pi} \sin^2 rt\,dt = \frac{1}{2}\int_0^{2\pi} (1 + \cos 2\,rt)dt$$

$$= \frac{1}{2}\left[t + \frac{\sin 2rt}{r}\right]_0^{2\pi}$$

$$= \frac{1}{2}\,[2\pi - 0] = \pi$$

$$C_r = \frac{\int_0^{2\pi} f(t)\sin rt}{\pi}$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} \sin rt\,dt - \int_{\pi}^{2\pi} \sin t + dt\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{-\cos rt}{r}\right)_0^{\pi} + \left(\frac{\cos rt}{r}\right)_{\pi}^{2\pi}\right] = \frac{2}{\pi\, r}(1 - \cos r\pi\ ) = \frac{2}{\pi\, r}[1 - (-1)^r\,]$$

For
$$r = 1, 2, 3$$

$$C_1 = \frac{2}{\pi}\,(1 - \cos\pi) = \frac{4}{\pi}$$

$$C_3 = \frac{2}{\pi \times 2}\,[1 - \cos 2\pi] = \frac{2}{2\pi}\,(1 - 1) = 0$$

$$C_3 = \frac{2}{\pi \times 3}\,[1 - \cos 3\pi] = \frac{2}{3\pi}\,[1 - (-1)] = \frac{4}{3\pi}$$

$$f(t) = \frac{4}{\pi}\,\sin t + \frac{4}{3\pi}\,\sin 3t$$

$\varepsilon = 0.099$ (as calculated above)

FOURIER SERIES

Representation of a function over a certain interval by a linear combination of mutually orthogonal functions is called fourier series representation.

Trigonometric Fourier Series

We know that functions sin $\omega_0 t$, sin $2\omega_0 t$ etc. form an orthogonal set over any interval $(t_0, t_0 + T)$. Similarly, function cos $n\omega_0 t$ is orthogonal to sin $m\omega_0 t$ over the same interval.

The complete set of functions consisting of a set cos $n\omega_0 t$ and sin $m\omega_0 t$ (for $n = 0, 1, 2, \ldots$) forms a complete orthogonal set. Thus any function $f(t)$ can be represented in terms of these functions over any interval $(t_0, t_0 + T)$.

$$f(t) = a_0 + a_1 \cos \omega_0 t + a_2 \cos 2\omega_0 t + \ldots + a_n \cos n\omega_0 t + \ldots$$
$$+ b_1 \sin \omega_0 t + b_2 \sin 2\omega_0 t + \ldots + b_n \sin n\omega_0 t$$

$$= a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \quad (t_0 \le t \le t_0 + T)$$

where
$$a_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \cos n\omega_0 t \, dt$$

$$b_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \sin n\omega_0 t \, dt$$

$$a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} f(t) dt$$

a_0 is the average value of $f(t)$ over the interval $(t_0, t_0 + T)$. Thus a_0 is the dc component of $f(t)$ over this interval.

Exponential Fourier Series

A set of exponential functions $\{e^{jn\omega_0 t}\}$, $n = 0, \pm 1, \pm 2, \ldots$ is orthogonal over an interval $(t_0, t_0 + T)$ for any value of t_o. It is therefore possible to represent an arbitrary function $f(t)$ by a linear combination of exponential functions over an interval $(t_0, t_0 + T)$

$$f(t) = F_0 + F_1 e^{j\omega_0 t} + F_2 e^{j2\omega_0 t} + \cdots + F_n e^{jn\omega_0 t} + \cdots$$
$$+ F_{-1} e^{-j\omega_0 t} + F_{-2} e^{-j2\omega_0 t} + \cdots + F_{-n} e^{-jn\omega_0 t} + \cdots$$

$$= \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \qquad (t_0 < t < t_0 + T)$$

where
$$\omega_0 = \frac{2\pi}{T}$$

Representation of $f(t)$ by exponential series is known as exponential fourier series representation of $f(t)$ over the interval $(t_0, t_0 + T)$. The coefficients in this series are given by,

$$F_n = \frac{1}{T} \int_{t}^{t_0} f(t) e^{-jn\omega_0 t} dt$$

Convergence of the Fourier Series (Dirichlet's Conditions)

The fourier series of function $f(t)$ exists only if the following conditions are satisfied:

1. The function $f(t)$ is a single valued function of the variable t within the interval (t_1, t_2).
2. The function $f(t)$ has a finite number of discontinuities in the interval (t_1, t_2).
3. The function $f(t)$ has a finite number of maxima and minima in the interval (t_1, t_2).
4. The function $f(t)$ is absolutely integrable i.e.

$$\int_{t_1}^{t_2} |f(t)| \, dt < \infty$$

These conditions are called as Dirichlet's conditions.

Fourier Series of Even and odd Function

We know that product of an even and odd function is an odd function and that product of an odd and odd function is an even function. Similarly the product of an even and even function is also an even function. For an even function

$$f_e(t)dt = \int_{-T}^{0} f_e(t)dt + \int_{0}^{T} f_e(t)\,dt$$

Letting $t = -x$

$$\therefore \quad \int_{-T}^{T} f_e(t)dt = \int_{0}^{T} f_e(-x)dx + \int_{0}^{T} f_e(t)dt$$

But $f_e(-x) = f_e(x)$

$$\therefore \quad \int_{-T}^{T} f_e(t)dt = \int_{0}^{T} f_e(x)dx + \int_{0}^{T} f_e(t)dt = 2\int_{0}^{T} f_e(t)dt$$

Similarly for an odd function $f_0(t)$

$$\int_{-T}^{T} f_0(t)dt = \int_{-T}^{T} f_0(t)dt + \int_{0}^{T} f_0(t)dt$$

$$= -\int_{0}^{T} f_0(x)dx + \int_{0}^{T} f_0(t)dt = 0$$

We know that

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos n\omega_0 t\,dt$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin n\omega_0 t\,dt$$

when $f(t)$ is an even function

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f_e(t)\cos n\omega_0 t\,dt$$

$$a_n = \frac{4}{T}\int_{0}^{T/2} f_e(t)\cos n\omega_0 t\,dt \qquad \left(\because f_e(t) \text{ and } \cos n\omega_0 t \text{ are even function}\right)$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f_e(t)\sin n\omega_0 t\,dt$$

$$= 0 \qquad (\because \sin n\omega_0 t \text{ is an odd } f_n)$$

when $f(t)$ is an odd function

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f_0(t)\cos n\omega_0 t\, dt$$

$$= 0 \qquad \text{(Product of odd and even function)}$$

$$b_n = \frac{4}{T}\int_{0}^{T/2} f_0(t)\sin n\omega_0 t\, dt$$

Thus a Fourier series of an even periodic function will consist entirely of cosine terms while a Fourier series of an odd periodic function will consists entirely of sine terms.

THE POWER SPECTRUM OF A PERIODIC FUNCTION

We know that exponential Fourier series of $f(t)$ is given by

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

The power associated with function $f(t)$ is

$$\frac{1}{T}\int_{-T/2}^{T/2} f^2(t)dt$$

$$\frac{1}{T}\int_{-T/2}^{T/2} f^2(t)dt = \frac{1}{T}\int_{-T/2}^{T/2} f(t)dt \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}dt$$

Interchanging the integration and summation on the R.H.S.,

$$\frac{1}{T}\int_{-T/2}^{T/2} f^2(t)dt = \frac{1}{T}\sum_{n=-\infty}^{\infty} F_n \int_{-T/2}^{T/2} f(t)e^{jn\omega_0 t}dt$$

$$= \frac{1}{T}\sum_{n=-\infty}^{\infty} F_n T F_{-n}$$

$$= \sum_{n=-\infty}^{\infty} F_n F_{-n}$$

Since $F_n = F^*_{-n}$, we have $F_n F_{-n} = |F_n|^2$

$$\frac{1}{T}\int_{-T/2}^{T/2} f^2(t)dt = \sum_{n=-\infty}^{\infty} |F_n|^2$$

$$= F_0^2 + |F_1|^2 + |F_2|^2 + \ldots + |F_n|^2 + \ldots + |F_{-1}|^2 + |F_{-2}|^2 + \ldots + |F_{-n}|^2 + \ldots$$

This is Parsval's theorem.

$$P = \frac{1}{T}\int_{-T/2}^{T/2} f^2(t)dt = \sum_{n=-\infty}^{\infty} F_n^2$$

Now, $|F_1|^2 = |F_{-1}|^2$

$$\begin{aligned} P &= F_0^2 + |F_1|^2 + |F_{-1}|^2 + |F_2|^2 + |F_{-2}|^2 + \dots \\ &= F_0^2 + 2|F_1|^2 + 2|F_2|^2 + \dots \\ &= F_0^2 + 2\sum_{n=1}^{\infty} |F_n|^2 \end{aligned}$$

The spectum drawn with the help of power of each component of the signal is called as power density spectrum.

Example 8 *Find exponential Fourier series of the waveform shown in Fig. 5.*

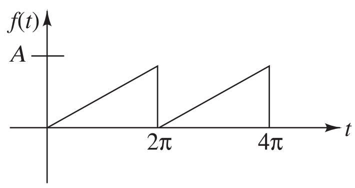


Fig. 5

Solution

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$

$$f(t) = \frac{A}{2\pi} t \qquad\qquad 0 < t < 2\pi$$

$$F_n = \frac{1}{2\pi} \int_0^{2\pi} \frac{A}{2\pi} t \cdot e^{-jnt} dt$$

$$= \frac{A}{4\pi^2} \left[t \cdot \frac{e^{-jnt}}{-jn} - \frac{e^{-jnt}}{(-jn)^2} \right]_0^{2\pi}$$

$$= \frac{A}{4\pi^2} \left[2\pi \cdot \frac{e^{-j2n\pi}}{-jn} + \frac{e^{-jn2\pi}}{n^2} - \frac{1}{n^2} \right]$$

$$= \frac{A}{4\pi^2} \left[\frac{2\pi}{-jn} + \frac{1}{n^2} - \frac{1}{n^2} \right]$$

$$= \frac{A}{4\pi^2} \frac{j2\pi}{n} = j\frac{A}{2\pi n}$$

$$F_0 = \frac{1}{2\pi} \int_0^{2\pi} \frac{A}{2\pi} t \cdot dt$$

$$= \frac{A}{4\pi^2} \left[\frac{t^2}{2} \right]_0^{2\pi} = \frac{A}{2}$$

$$\therefore \qquad f(t) = \frac{A}{2} + \sum_{\substack{n=1 \\ n=\text{odd}}}^{\infty} j\frac{A}{2\pi n} e^{jn\omega_0 t}$$

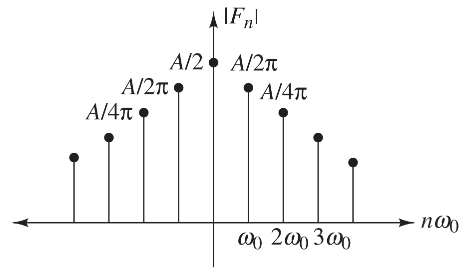


Fig. 6

Example 9 *Expand periodic gate function by the exponential Fourier series.*

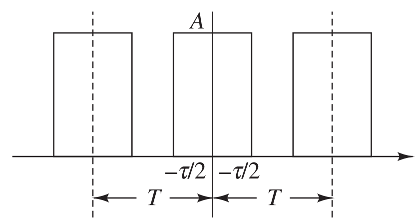


Fig. 7

Solution

$$f(t) = A \qquad (-\tau/2 < t < \tau/2)$$
$$= 0 \qquad (\tau/2 < t < T - \tau)$$

$$F_n = \frac{1}{T}\int_{-\tau/2}^{\tau/2} f(t)e^{-jn\omega_0 t}\,dt$$

$$= \frac{1}{T}\int_{-\tau/2}^{\tau/2} Ae^{-jn\omega_0 t}\,dt$$

$$= \frac{A}{T}\left[\frac{e^{-jn\omega_0 t}}{-jn\omega_0}\right]_{-\tau/2}^{\tau/2}$$

$$= \frac{A}{T}\left[\frac{e^{-jn\omega_0\tau/2} - e^{jn\omega_0\tau/2}}{-jn\omega_0}\right]$$

$$= \frac{2A}{n\omega_0 T}\sin n\omega_0\tau/2$$

$$= \frac{A\tau}{T}\left[\frac{\sin(n\omega_0\tau/2)}{n\omega_0 T/2}\right]$$

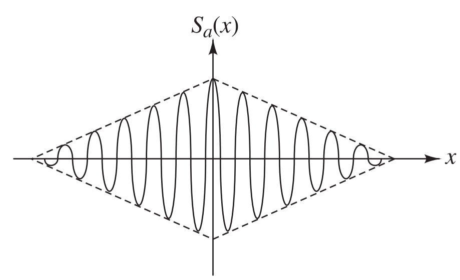


Fig. 8

$$= \frac{A\tau}{T} S_a(n\omega_0 \tau/2) \qquad \because \left(\omega_0 = \frac{2\pi}{T}\right)$$

$$= \frac{A\tau}{T} S_a\left(\frac{n\pi\tau}{T}\right)$$

$$\therefore \qquad f(t) = \frac{A\tau}{T} \sum_{n=-\infty}^{\infty} S_a\left(\frac{n\pi\tau}{T}\right) e^{jn\omega_0 t}$$

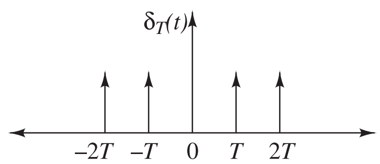


Fig. 9

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Let $$\delta_T(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

$$F_n = \frac{1}{T} \int_{-T/2}^{T/2} \delta_T(t) e^{-jn\omega_0 t} dt$$

$$= \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-jn\omega_0 t} dt$$

$$= \frac{1}{T} \; \delta(0)\, e^0 = \frac{1}{T}$$

$$\therefore \qquad \delta_T(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T} e^{jn\omega_0 t}$$

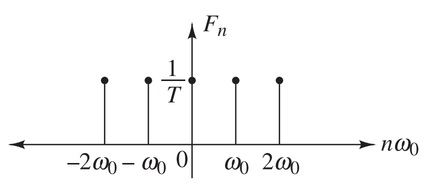


Fig. 10

Example 10 *Obtain trigonometric Fourier series representation of the waveforms shown or show that the function shown can be represented by orthogonal set in the interval $0 < t < 2\pi$.*

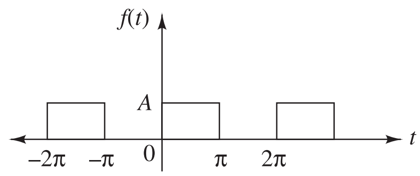


Fig. 11

Solution

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$

$$f(t) = A \qquad 0 < t < \pi$$

$$= 0 \qquad \pi < t < 2\pi$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

$$a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)dt$$

$$= \frac{1}{2\pi}\int_0^{\pi} A dt = \frac{A}{2\pi}[t]_0^{\pi} = \frac{A}{2\pi}[\pi - 0] = \frac{A}{2}$$

$$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos n\omega_0 t\, dt$$

$$= \frac{2}{2\pi}\int_0^{\pi} A\cos nt\, dt$$

$$= \frac{A}{\pi}\left[\frac{\sin nt}{n}\right]_0^{\pi} = 0 \qquad \{\because \sin n\pi = \sin 0 = 0\}$$

$$b_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin n\omega_0 t\, dt$$

$$= \frac{2}{2\pi}\int_0^{\pi} A\sin nt\, dt$$

$$= \frac{A}{\pi}\left[-\frac{\cos nt}{n}\right]_0^{\pi} = \frac{A}{n\pi}[-\cos n\pi + \cos 0] \qquad \left\{\begin{matrix}\cos n\pi = (-1)^n \\ \cos 0 = 1\end{matrix}\right\}$$

$$= \frac{A}{n\pi}[1 - (-1)^n]$$

$$= \frac{2A}{n\pi} \qquad \text{if } n \text{ is odd}$$

$$= 0 \qquad \text{if } n \text{ is even}$$

$$f(t) = \frac{A}{2} + \sum_{\substack{n=1\\ n=\text{odd}}}^{\infty} \frac{2A}{n\pi} \sin n\omega_0 t$$

$$= \frac{A}{2} + \frac{2A}{\pi} \sin \omega_0 t + \frac{2A}{3\pi} \sin 3\omega_0 t + \ldots$$

$$C_n = \sqrt{a_n^2 + b_n^2}$$

$$\phi_n = -\tan^{-1}\left(\frac{b_n}{a_n}\right)$$

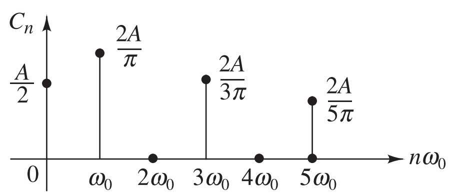


Fig. 12

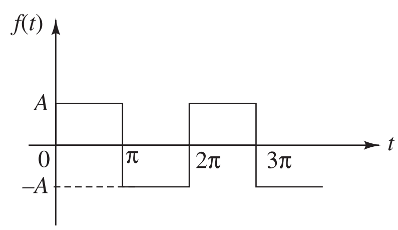


Fig. 13

Solution

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$

$$f(t) = A \qquad 0 < t < \pi$$

$$= -A \qquad \pi < t < 2\pi$$

$$a_0 = \frac{1}{2\pi}\left[\int_0^{\pi} A\,dt + \int_{\pi}^{2\pi} (-A)\,dt\right]$$

$$= \frac{A}{2\pi}\left[(t)_0^{\pi} + (-t)_{\pi}^{2\pi}\right]$$

$$= \frac{1}{2\pi}\,[\pi - 0 - 2\pi + \pi] = 0$$

$$a_n = \frac{2}{2\pi}\left[\int_0^{\pi} A\cos n\omega_0 t\,dt + \int_{\pi}^{2\pi} (-A)\cos n\omega_0 t\,dt\right]$$

$$= \frac{A}{\pi}\left[\left(\frac{\sin nt}{n}\right)_0^{\pi} - \left(\frac{\sin nt}{n}\right)_{\pi}^{2\pi}\right]$$

$$= \frac{A}{n\pi} \qquad [\sin n\pi - \sin 0 - \sin 2n\pi + \sin n\pi]$$

$$= 0 \qquad \{\sin n\pi = \sin 2n\pi = \sin 0 = 0\}$$

$$b_n = \frac{2}{2\pi}\left[\int_0^{\pi} A\sin n\omega_0 t\,dt + \int_{\pi}^{2\pi} (-A)\sin n\omega_0 t\,dt\right]$$

$$= \frac{A}{n\pi}\left[\left(-\frac{\cos nt}{n}\right)_0^{\pi} + \left(\frac{\cos nt}{n}\right)_{\pi}^{2\pi}\right]$$

$$= \frac{A}{n}\,[-(-1)^n + 1 + 1 - (-1)^n]$$

$$= \frac{2A}{n\pi}\,[1 - (-1)^n] \qquad \left\{\begin{matrix}\cos n\pi = (-1)^n \\ \cos 2n\pi = 1\end{matrix}\right\}$$

$$= \frac{4A}{n\pi} \qquad \text{if } n \text{ is odd}$$

$$= 0 \qquad \text{if } n \text{ is even}$$

$$f(t) = \sum_{\substack{n=1 \\ n=\text{odd}}}^{\infty} \frac{4A}{n\pi}\sin n\omega_0 t$$

$$= \frac{4A}{\pi}\sin \omega_0 t + \frac{4A}{3\pi}\sin 3\omega_0 t + \ldots$$

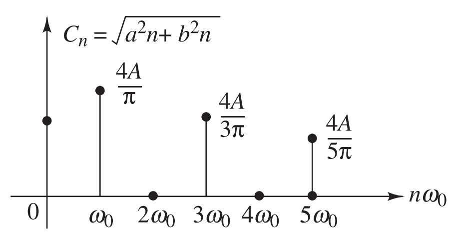


Fig. 14

(*iii*)

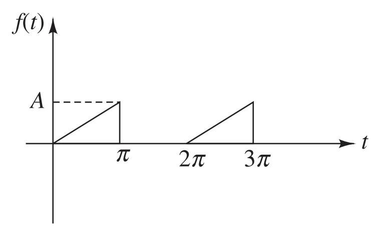


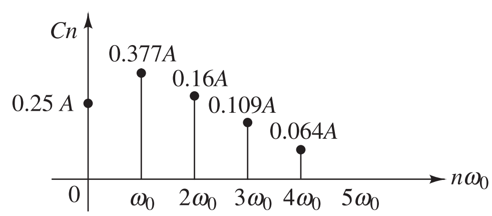
Fig. 15

$$\omega_0 = \frac{2\pi}{2\pi} = 1$$

$$f(t) = \frac{A}{\pi} t \qquad 0 < t < \pi$$

$$= 0 \qquad \pi < t < 2\pi$$

$$a_0 = \frac{1}{2\pi} \left[\int_0^{\pi} \frac{A}{\pi} t \, dt \right]$$



Fi. 16

$$= \frac{A}{2\pi^2} \left[\left(\frac{t^2}{2} \right)_0^{\pi} \right]$$

$$= \frac{A}{2\pi^2} \left[\frac{\pi^2}{2} - 0 \right] = \frac{A}{4}$$

$$a_n = \frac{1}{\pi} \left[\int_0^{\pi} \frac{A}{\pi} t \cos n\omega_0 t \, dt \right]$$

$$= \frac{A}{\pi^2} \left[\int_0^{\pi} t \cos nt \, dt \right]$$

$$= \frac{A}{\pi^2} \left[t \cdot \frac{\sin nt}{n} - \int \frac{\sin nt}{n} \cdot 1 dt \right]_0^{\pi}$$

$$= \frac{A}{\pi^2}\left[t \cdot \frac{\sin nt}{n} + \frac{1}{n^2}\cos nt\right]_0^{\pi}$$

$$= \frac{A}{\pi^2}\left[\frac{1}{n^2}\{(-1)^n - 1\}\right]$$

$$= \frac{-2A}{n^2\pi^2} \qquad \text{if } n \text{ is odd}$$

$$= 0 \qquad \text{if } n \text{ is even}$$

$$b_n = \frac{1}{\pi}\left[\int_0^{\pi} \frac{A}{\pi} t \sin n\omega_0 t\, dt\right]$$

$$= \frac{A}{\pi^2}\left[\int_0^{\pi} t \sin t\, dt\right]$$

$$= \frac{A}{\pi^2}\left[t \cdot \frac{(-\cos nt)}{n} + \int \frac{\cos nt}{n} \cdot dt\right]_0^{\pi}$$

$$= \frac{A}{\pi^2}\left[-t \cdot \frac{\cos nt}{n} + \frac{1}{n^2}\sin nt\right]_0^{\pi}$$

$$= \frac{-A}{n\pi^2}\,[\pi(-1)^n - 0]$$

$$= \frac{A}{n\pi^2}\,(-1)^{n+1} = \frac{A}{n\pi} \qquad n \text{ is odd}$$

$$= -\frac{A}{n\pi} \qquad n \text{ is even}$$

$$f(t) = \frac{A}{4} + \sum_{\substack{n=1\\ n=\text{odd}}}^{\infty} -\frac{2A}{n^2\pi^2}\cos nt + \sum_{n=1}^{\infty} \frac{A}{n\pi}(-1)^{n+1}\sin nt$$

$$C_n = \sqrt{a_n^2 + b_n^2} \qquad C_n = \frac{V}{n\pi} \qquad n \text{ is even}$$

$$= \sqrt{a_n^2 + b_n^2} \qquad n \text{ is odd}$$